This sample exam is provided to serve as one component of your studying for this exam in this course. Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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1. [12 points] Six matrices and their eigenvalues and eigenvectors are given below. Use this information to answer the questions below. Be sure that you explain your answers.

<table>
<thead>
<tr>
<th>$\mathbf{A}_1$</th>
<th>$\mathbf{A}_2$</th>
<th>$\mathbf{A}_3$</th>
<th>$\mathbf{A}_4$</th>
<th>$\mathbf{A}_5$</th>
<th>$\mathbf{A}_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{pmatrix} -1 &amp; 2 \ -1 &amp; -3 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 2 &amp; 2 \ 1 &amp; 3 \end{pmatrix}$</td>
<td>$\begin{pmatrix} -2 &amp; 2 \ 1 &amp; -3 \end{pmatrix}$</td>
<td>$\begin{pmatrix} -1 &amp; 3 \ 2 &amp; -2 \end{pmatrix}$</td>
<td>$\begin{pmatrix} -2 &amp; -2 \ -1 &amp; -3 \end{pmatrix}$</td>
<td>$\begin{pmatrix} -3 &amp; -1 \ 1 &amp; -1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$\lambda_{1,2} = -2 \pm i$</td>
<td>$\lambda_{1,2} = 1,4$</td>
<td>$\lambda_{1,2} = -4,-1$</td>
<td>$\lambda_{1,2} = -4,1$</td>
<td>$\lambda_{1,2} = -4,-1$</td>
<td>$\lambda_{1,2} = -2,-2$</td>
</tr>
<tr>
<td>$\mathbf{v}_1 = \begin{pmatrix} 2 \ -1 + i \end{pmatrix}$</td>
<td>$\mathbf{v}_1 = \begin{pmatrix} -2 \ 1 \end{pmatrix}$</td>
<td>$\mathbf{v}_1 = \begin{pmatrix} -1 \ 1 \end{pmatrix}$</td>
<td>$\mathbf{v}_1 = \begin{pmatrix} 1 \ 1 \end{pmatrix}$</td>
<td>$\mathbf{v}_1 = \begin{pmatrix} -1 \ 1 \end{pmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\mathbf{v}_2 = \begin{pmatrix} 2 \ -1 - i \end{pmatrix}$</td>
<td>$\mathbf{v}_2 = \begin{pmatrix} 1 \ 1 \end{pmatrix}$</td>
<td>$\mathbf{v}_2 = \begin{pmatrix} 2 \ 1 \end{pmatrix}$</td>
<td>$\mathbf{v}_2 = \begin{pmatrix} 3 \ 2 \end{pmatrix}$</td>
<td>$\mathbf{v}_2 = \begin{pmatrix} -2 \ 1 \end{pmatrix}$</td>
<td>$\mathbf{w} = \begin{pmatrix} 1 \ 0 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

a. [6 points] Write a linear system involving one of the $\mathbf{A}_j$ that could have the phase portrait shown to the right.

b. [6 points] Write a linear system involving one of the $\mathbf{A}_j$ that could have the phase portrait shown to the right.
2. [12 points] For each of the following, give an example, as indicated.
   
   a. [4 points] Give a first-order differential equation that could have the phase line shown to the right.

   ![Phase Line](image)

   b. [4 points] Give a second-order, linear, constant-coefficient, nonhomogeneous differential equation that could have the response shown to the right.

   ![Response Graph](image)

   c. [4 points] Give a system of two linear, first-order, constant-coefficient differential equations which have an isolated critical point at the origin that is an unstable saddle point.
3. [12 points] Suppose a model for a physical system (e.g., a circuit or a mass-spring system) is given by the differential equation $L[y] = y'' + ay' + by = k$ (where $a$, $b$, and $k$ are real numbers).

a. [4 points] If the solution to the problem with some initial conditions is $y = e^{-t} \cos(2t) - e^{-t} \sin(2t) + 2$, what can you say about $a$, $b$, and $k$?

b. [4 points] If the solution to the problem with some initial conditions is $y = e^{-t} \cos(2t) - e^{-t} \sin(2t) + 2$, sketch a phase portrait for the system. Be sure it is clear how you obtain your solution.

c. [4 points] Now suppose that the solution to the problem with some initial conditions is $y = e^{-t} \cos(2t) - e^{-t} \sin(2t) + 2$, and that at some time $t = t_0$ we remove the forcing term ($k$). Write a single differential equation you could solve to find $y$ for all $t \geq 0$. What initial conditions apply at $t = 0$?
4. [12 points] Consider the predator-prey model with harvesting (harvesting here implies hunting by humans, e.g., fishing if the populations are fish) given by

\[ x' = x(3 - x - y) - 2, \quad y' = y(-3 + x). \]

Note that as \( x \) and \( y \) are populations, we must have \( x, y \geq 0 \).

a. [3 points] Explain what each term in the equation for \( x \) models. Is \( x \) or \( y \) the predator? Which population is being harvested?

b. [7 points] By doing an appropriate linear analysis, sketch a phase portrait for this system.

c. [2 points] Based on your answer to (b), sketch what you expect the behavior of the solution to the system will be as a function of time if \( x(0) = 3 \) and \( y(0) = 1 \). How would you expect this to differ from the behavior with the initial condition \( x(0) = 1, y(0) = 1 \)?
5. [10 points] Consider the linear system

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -1 & 0 & \alpha^2 \\ 0 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]

a. [5 points] For what values of \( \alpha \), if any, will all solutions to the system remain bounded as \( t \to \infty \)?

b. [5 points] Now suppose that \( \alpha = 2 \). Are there any initial conditions for which solutions to the system will remain bounded? If so, what are they? Explain.

\[1\text{ Possibly useful: } \det\left( \begin{pmatrix} a & 0 & b \\ 0 & c & d \\ e & 0 & f \end{pmatrix} \right) = acf - bce.\]
6. [12 points] In lab we considered an electrical system $y'' + \omega_0^2 y = f(t)$ which produced a response similar to that shown in the figure to the right. In this problem, we will take $\omega_0 = \pi$, and use the initial conditions $y(0) = 0$, $y'(0) = 1$.

a. [6 points] If we pick $f(t) = k \delta(t - t_0)$, what is $t_0$? Solve the problem with this $f(t)$ to find a value of $k$ that results in a solution that could produce a graph similar to this one. Explain your logic.

b. [6 points] Now suppose $f(t) = k I(t)$, where $I(t)$ is the finite-width impulse we used in lab 4. Let $I(t)$ have height 8 and width $\frac{1}{8} = 0.125$, applied at $t = 2$ (that is, $I(t) = 8$ for $2 < t < 2.125$, and is zero elsewhere). Solve the resulting problem to find $y$. What should $k$ be to produce a solution similar to that graphed above (your work in lab may be useful here)? If you used this value of $k$ in (a), how would the response graph be different?
7. [15 points] DO complete this problem if you have NOT completed the mastery assessment. DO NOT complete it if you have completed the mastery assessment. Find explicit, real-valued solutions for each of the following.

a. [7 points] Find $Q(z)$ if $(z + 1)Q' = 3Q^2$, $Q(0) = 4$.

b. [8 points] Find the general solution to the first-order system \[
\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 13 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]
8. [15 points] DO complete this problem if you have NOT completed the mastery assessment. DO NOT complete it if you have completed the mastery assessment. Find explicit, real-valued solutions for each of the following.

a. [7 points] Find \( W(t) \) if \( W'' - 2W' - 8W = 24t + 54, W(0) = 0, W'(0) = 0. \)

b. [8 points] Find \( p(t) \) if \( \delta(t - 2), p(0) = 0, p'(0) = 9. \)
This page provided for additional work.
Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:
  
  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
  
  $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
  
  $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
  
  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

  About $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$.

- Some integration formulas:
  
  $\int uv' dt = uv - \int u'v dt$; thus $\int te^t dt = te^t - e^t + C$; $\int t \cos(t) dt = t \sin(t) + \cos(t) + C$, and $\int t \sin(t) dt = -t \cos(t) + \sin(t) + C$.

- Euler’s formula: $e^{i\theta} = \cos \theta + i \sin \theta$.

- A coarse summary of partial fractions:

  $$\frac{1}{(s + r_1)(s + r_2)^2((s + h)^2 + k^2)} = \frac{A}{s + r_1} + \frac{B}{s + r_2} + \frac{C}{(s + r_2)^2} + \frac{D(s + h) + E}{(s + h)^2 + k^2}.$$  

Some Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$\frac{1}{s}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$, $s &gt; a$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>$\sin(at)$</td>
<td>$\frac{a}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$\cos(at)$</td>
<td>$\frac{s}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$u_c(t)$</td>
<td>$\frac{e^{-cs}}{s}$</td>
</tr>
<tr>
<td>$\delta(t - c)$</td>
<td>$e^{-cs}$</td>
</tr>
</tbody>
</table>

A. $f'(t)$ | $sF(s) - f(0)$ |
A.1 $f''(t)$ | $s^2F(s) - s f(0) - f'(0)$ |
A.2 $f^{(n)}(t)$ | $s^nF(s) - \ldots - f^{(n-1)}(0)$ |
B. $t^n f(t)$ | $(-1)^n F^{(n)}(s)$ |
C. $e^{ct} f(t)$ | $F(s - c)$ |
D. $u_c(t) f(t - c)$ | $e^{-cs} F(s)$ |
E. $f(t)$ (periodic with period $T$) | $\frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t) \, dt$ |
F. $\int_0^t f(x) g(t - x) \, dx$ | $F(s)G(s)$ |