Math 216 — Final Exam

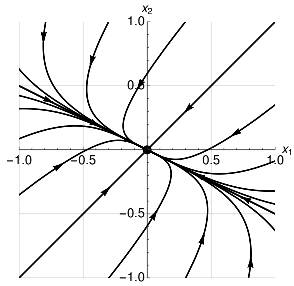
26 April, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

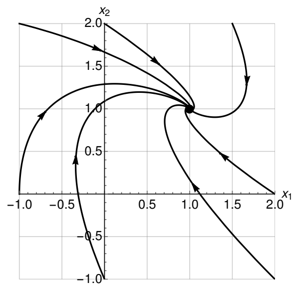
1. [12 points] Six matrices and their eigenvalues and eigenvectors are given below. Use this information to answer the questions below. Be sure that you explain your answers.

$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{A}_5$	$\mathbf{A}_6$
$\begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}$	$\begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix}$	$\begin{pmatrix} -2 & -2 \\ -1 & -3 \end{pmatrix}$	$\begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}$
$\lambda_{1,2} = -2 \pm i$	$\lambda_{1,2} = 1, 4$	$\lambda_{1,2} = -4, -1$	$\lambda_{1,2} = -4, 1$	$\lambda_{1,2} = -4, -1$	$\lambda_{1,2} = -2, -2$
$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1+i \end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
$\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1-i \end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$	$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

a. [6 points] Write a linear system involving one of the $\mathbf{A}_j$ that could have the phase portrait shown to the right.

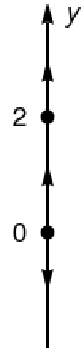


b. [6 points] Write a linear system involving one of the $\mathbf{A}_j$ that could have the phase portrait shown to the right.

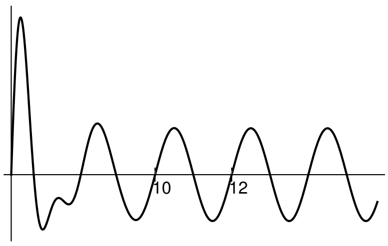


2. [12 points] For each of the following, give an example, as indicated.

a. [4 points] Give a first-order differential equation that could have the phase line shown to the right.



b. [4 points] Give a second-order, linear, constant-coefficient, nonhomogeneous differential equation that could have the response shown to the right.



c. [4 points] Give a system of two linear, first-order, constant-coefficient differential equations which have an isolated critical point at the origin that is an unstable saddle point.

3. [12 points] Suppose a model for a physical system (e.g., a circuit or a mass-spring system) is given by the differential equation $L[y] = y'' + ay' + by = k$ (where a, b, and k are real numbers).

a. [4 points] If the solution to the problem with some initial conditions is $y = e^{-t}\cos(2t) - e^{-t}\sin(2t) + 2$, what can you say about a, b, and k?

b. [4 points] If the solution to the problem with some initial conditions is $y = e^{-t}\cos(2t) - e^{-t}\sin(2t) + 2$, sketch a phase portrait for the system. Be sure it is clear how you obtain your solution.

c. [4 points] Now suppose that the solution to the problem with some initial conditions is $y = e^{-t}\cos(2t) - e^{-t}\sin(2t) + 2$, and that at some time $t = t_0$ we remove the forcing term (k). Write a single differential equation you could solve to find y for all $t \geq 0$. What initial conditions apply at $t = 0$?

4. [12 points] Consider the predator-prey model with harvesting (harvesting here implies hunting by humans, e.g., fishing if the populations are fish) given by

$$x' = x\,(3 - x - y) - 2, \quad y' = y\,(-3 + x)\,.$$

Note that as x and y are populations, we must have $x, y \geq 0$.

a. [3 points] Explain what each term in the equation for x models. Is x or y the predator? Which population is being harvested?

b. [7 points] By doing an appropriate linear analysis, sketch a phase portrait for this system.

c. [2 points] Based on your answer to (b), sketch what you expect the behavior of the solution to the system will be as a function of time if $x(0) = 3$ and $y(0) = 1$. How would you expect this to differ from the behavior with the initial condition $x(0) = 1$, $y(0) = 1$?

5. [10 points] Consider the linear system

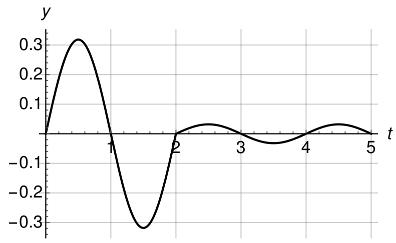
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -1 & 0 & \alpha^2 \\ 0 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

a. [5 points] For what values of α, if any, will all solutions to the system remain bounded as $t \to \infty$?[1]

b. [5 points] Now suppose that $\alpha = 2$. Are there any initial conditions for which solutions to the system will remain bounded? If so, what are they? Explain.

[1] *Possibly useful:* $\det\left(\begin{pmatrix} a & 0 & b \\ 0 & c & d \\ e & 0 & f \end{pmatrix}\right) = acf - bce.$

6. [12 points] In lab we considered an electrical system $y'' + \omega_0^2 y = f(t)$ which produced a response similar to that shown in the figure to the right. In this problem, we will take $\omega_0 = \pi$, and use the initial conditions $y(0) = 0$, $y'(0) = 1$.



a. [6 points] If we pick $f(t) = k\delta(t - t_0)$, what is t_0? Solve the problem with this $f(t)$ to find a value of k that results in a solution that could produce a graph similar to this one. Explain your logic.

b. [6 points] Now suppose $f(t) = kI(t)$, where $I(t)$ is the finite-width impulse we used in lab 4. Let $I(t)$ have height 8 and width $\frac{1}{8} = 0.125$, applied at $t = 2$ (that is, $I(t) = 8$ for $2 < t < 2.125$, and is zero elsewhere). Solve the resulting problem to find y. What should k be to produce a solution similar to that graphed above (your work in lab may be useful here)? If you used this value of k in (a), how would the response graph be different?

7. [15 points] **DO** complete this problem if you have **NOT** completed the mastery assessment. **DO NOT** complete it if you have completed the mastery assessment. Find explicit, real-valued solutions for each of the following.

a. [7 points] Find $Q(z)$ if $(z+1)Q' = 3Q^2$, $Q(0) = 4$.

b. [8 points] Find the general solution to the first-order system $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 13 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

8. [15 points] **DO** complete this problem if you have **NOT** completed the mastery assessment. **DO NOT** complete it if you have completed the mastery assessment. Find explicit, real-valued solutions for each of the following.

a. [7 points] Find $W(t)$ if $W'' - 2W' - 8W = 24t + 54$, $W(0) = 0$, $W'(0) = 0$.

b. [8 points] Find $p(t)$ if $p'' + 8p' + 16p = 6\delta(t-2)$, $p(0) = 0$, $p'(0) = 9$.

This page provided for additional work.

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:
  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ About $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\,v'\,dt = u\,v - \int u'\,v\,dt$; thus $\int t\,e^t\,dt = t\,e^t - e^t + C$, $\int t\,\cos(t)\,dt = t\,\sin(t) + \cos(t) + C$, and $\int t\,\sin(t)\,dt = -t\,\cos(t) + \sin(t) + C$.
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.
- A coarse summary of partial fractions:

$$\frac{1}{(s+r_1)(s+r_2)^2((s+h)^2+k^2)} = \frac{A}{s+r_1} + \frac{B}{s+r_2} + \frac{C}{(s+r_2)^2} + \frac{D(s+h)+E}{(s+h)^2+k^2}.$$

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta(t-c)$	e^{-cs}
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^nF(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{c\,t} f(t)$	$F(s-c)$
D.	$u_c(t)\,f(t-c)$	$e^{-cs}\,F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$
F.	$\int_0^t f(x)g(t-x)\,dx$	$F(s)G(s)$