Math 216 — First Midterm

20 February, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Solve each of the following, finding explicit real-valued solutions as indicated.

a. [7 points] Find the general solution to $y' = \dfrac{5 + 5s^5 - 5s^4 y}{1 + s^5}$.

Solution: Simplifying the fraction on the right-hand side, this is $y' = 5 - \frac{5s^4}{1+s^5}\, y$, which is a first-order linear problem. In standard form, this is $y' + \frac{5s^4}{1+s^5}\, y = 5$, so (noting that $\int \frac{5s^4}{1+s^5}\, ds = \ln|1 + s^5|$) an integrating factor is $\mu = 1 + s^5$. Multiplying both sides by μ, $(\mu\, y)' = 5 + 5s^5$. Integrating, $(1 + s^5)y = 5s + \frac{5}{6}s^6 + C$, so that

$$y = \frac{5s + \frac{5}{6}s^6 + C}{1 + s^5}.$$

b. [8 points] Solve the initial value problem $R' = (2 - 10z)R^2$, $R(0) = -2$.

Solution: This is first-order and nonlinear, but separable. Separating, we have $R'/R^2 = 2 - 10z$, so that $-R^{-1} = 2z - 5z^2 + C$, and

$$R = -\frac{1}{2z - 5z^2 + C}.$$

For $R(0) = -2$, $C = \frac{1}{2}$, and

$$R = -\frac{1}{2z - 5z^2 + \frac{1}{2}}.$$

2. [15 points] Solve each, finding explicit real-valued solutions as indicated.

a. [8 points] Solve the initial value problem $x' = -y$, $y' = 12x - 7y$, $x(0) = 2$, $y(0) = 1$.

Solution: In matrix form, this is $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 12 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. The eigenvalues of the coefficient matrix are given by $\det(\begin{pmatrix} -\lambda & -1 \\ 12 & -7-\lambda \end{pmatrix}) = \lambda^2 + 7\lambda + 12 = (\lambda+3)(\lambda+4) = 0$. Thus $\lambda = -4$ or $\lambda = -3$. Note that the first row of the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ for the eigenvector gives $\mathbf{v} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$, so the corresponding eigenvectors are $\mathbf{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. The general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t}.$$

Applying the initial conditions, we have $c_1 + c_2 = 2$ and $4c_1 + 3c_2 = 1$. Subtracting the second from four times the first, $c_2 = 7$, so that $c_1 = -5$. The solution is

$$\mathbf{x} = -5 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-4t} + 7 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t}.$$

b. [7 points] Find the general solution to $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 6 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Solution: The eigenvalues of the coefficient matrix are given by $(6-\lambda)(-2-\lambda) + 20 = \lambda^2 - 4\lambda + 8 = (\lambda - 2)^2 + 4 = 0$. Thus $\lambda = 2 \pm 2i$. If $\lambda = 2 + 2i$, the components of the eigenvector satisfy $(4 - 2i)v_1 - 5v_2 = 0$, so we may take $\mathbf{v} = \begin{pmatrix} 5 \\ 4 - 2i \end{pmatrix}$. A complex-valued solution is therefore $\mathbf{x} = \begin{pmatrix} 5 \\ 4 - 2i \end{pmatrix} e^{2t}(\cos(2t) + i\sin(2t))$. Separating the real and imaginary parts of this, we have

$$\mathbf{x} = c_1 \begin{pmatrix} 5\cos(2t) \\ 4\cos(2t) + 2\sin(2t) \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 5\sin(2t) \\ -2\cos(2t) + 4\sin(2t) \end{pmatrix} e^{2t}.$$

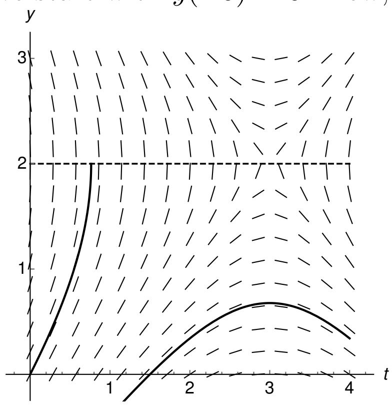
Alternately, we could take $\mathbf{v} = \begin{pmatrix} 2+i \\ 2 \end{pmatrix}$, so that

$$\mathbf{x} = c_1 \begin{pmatrix} 2\cos(2t) - \sin(2t) \\ 2\cos(2t) \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} \cos(2t) + 2\sin(2t) \\ 2\sin(2t) \end{pmatrix} e^{2t}.$$

3. [12 points] Suppose we are solving the initial value problem $y' = \dfrac{t-3}{y-2}$, $y(0) = y_0$.

a. [6 points] A direction field for the differential equation is shown to the right, below. Using this and your knowledge of the differential equation, explain what the solution will look like if we start with the initial condition $y(0) = 0$, and if we start with $y(1.5) = 0$. How, and why, are these solutions different?

(The printed exam copy had $y(1) = 0$ for the second initial condition. This was supposed to be $y(1.5)$; through $(0, 1)$ the solution is $y = t - 1$.)



Solution: With $y(0) = 0$, we expect the solution that bends up from $(0, 0)$ until it gets to $y = 2$. At $y = 2$, the right-hand side of the differential equation becomes undefined and we expect that we may have trouble continuing the solution. In this case it appears that the solution tries to bend back on itself, which it cannot do. Therefore, we expect that at $y = 2$ we expect the solution to end. This makes sense, because we would anticipate that any initial condition $y(t_0) = 2$ may not have a solution, because of the existence and uniqueness theorem.

From $y(1.5) = 0$, the solution appears to grow and turn over, then continuing to larger negative values. Thus y never reaches $y = 2$, and we expect the solution to exist for all times.

b. [6 points] Solve the problem with initial condition $y(0) = 0$. Based on your solution, for what values of t and y does your solution exist? How is this related to the existence and uniqueness theorem?

Solution: Separating values and integrating, we have $\frac{1}{2}y^2 - 2y = \frac{1}{2}t^2 - 3t + C$, and the initial condition $y(0) = 0$ requires that $C = 0$. We can find an explicit solution for y by multiplying by 2 and using the quadratic formula: $y^2 - 4y - t^2 + 6t = 0$, so $y = 2 \pm \sqrt{4 - (-t^2 + 6t)}$. To have $y(0) = 0$ we take the negative, so $y = 2 - \sqrt{t^2 - 6t + 4}$. This will work until $y = 0$, which is when $t^2 - 6t + 4 = (t - 3)^2 - 5 = 0$, so $t = 3 \pm \sqrt{5}$. Thus we expect the solution to exist for $0 \le t < 3 - \sqrt{5}$, $0 \le y < 2$.

At $y = 2$, the right-hand side of the equation become discontinuous, and the existence and uniqueness theorem doesn't guarantee a solution through the initial condition $y(3 - \sqrt{5}) = 2$. Thus we may expect the solution to break down there.

4. [15 points] Consider the system of differential equations given by $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ with the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

a. [4 points] If $\mathbf{P}(t) = \begin{pmatrix} 0 & 1 \\ -2t^{-2} & 2t^{-1} \end{pmatrix}$, is this a linear or nonlinear problem? If we apply the initial condition, will there be a unique solution? Explain.

Solution: This is a linear problem, though non-constant coefficient. Accordingly, there will be a unique solution through any $\mathbf{x}(t_0) = \mathbf{x}_0$ where $\mathbf{P}(t)$ is continuous. That is, through any $t_0 \neq 0$. The solution will exist on the interval $(0, \infty)$ or $(-\infty, 0)$, depending on whether $t_0 > 0$ or $t_0 < 0$.

b. [6 points] If $\mathbf{P}(t) = \mathbf{A}$, a 2×2 constant real-valued matrix, and if a general solution to the system is $\mathbf{x} = c_1\mathbf{v}_1e^{\lambda t} + c_2(t\mathbf{v}_1 + \mathbf{v}_2)e^{\lambda t}$, how many solutions are there to each of the following algebraic systems of equations? Why?
(i) $\mathbf{A}\mathbf{x} = 2\lambda\mathbf{x}$
(ii) $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{v}_1$

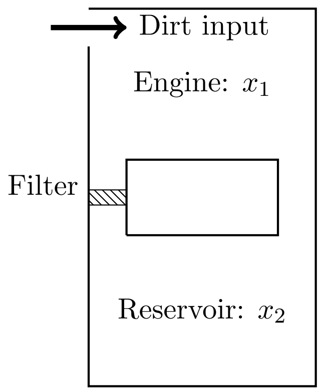
Solution: (i) Note that we know the only eigenvalue is λ, with eigenvector $\mathbf{v}_1$. Thus 2λ is not an eigenvalue, and we cannot find a non-zero solution to $\mathbf{A}\mathbf{x} = 2\lambda\mathbf{x}$. The only solution is $\mathbf{x} = \mathbf{0}$.
(ii) In this case, we know there are an infinite number of solutions: we're solving for the generalized eigenvector, which is only unique up to an additive multiple of the eigenvector $\mathbf{v}_1$. We can see this directly by noting that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = 0$: thus

$$(\mathbf{A} - \lambda\mathbf{I})(\mathbf{v}_2 + k\mathbf{v}_1) = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 + k(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{v}_1 + \mathbf{0}.$$

c. [5 points] If $\mathbf{P}(t) = \mathbf{B}$, a 2×2 constant real-valued matrix, and a solution to the system is $\mathbf{x} = \begin{pmatrix} \cos(3t) \\ \cos(3t) - 2\sin(3t) \end{pmatrix} e^{-4t}$, what are the eigenvalues and eigenvectors of $\mathbf{B}$?

Solution: If this is a solution, we can immediately see that the eigenvalues must be $\lambda = -4 \pm 3i$, because the time dependence of the solution comes from $e^{\lambda t} = e^{(\mu+\nu)t} = e^{\mu t}(\cos(\nu t) + i\sin(\nu t))$. We may then guess that $\mathbf{x}$ is the real or imaginary part of $\mathbf{v}e^{\lambda t}$, where $\mathbf{v}$ is the corresponding eigenvector. This leads us to conclude that $\mathbf{v} = \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix}$, if $\mathbf{x}$ is the real part, or $\mathbf{v} = \begin{pmatrix} i \\ -2 + i \end{pmatrix}$, if $\mathbf{x}$ is the imaginary part.

5. [16 points] In internal combustion engines, oil is circulated from a reservoir, around moving parts to lubricate them, and back to the reservoir. As it circulates, it collects dirt from the engine. To remove the dirt, oil from the reservoir is passed through a filter. A simple model for this system is shown to the right. Dirt is "added" to the oil in the engine, and we denote the amount of dirt in the engine compartment as x_1 and that in the reservoir as x_2. Suppose that the amount of oil in the engine compartment is 3 quarts and in the reservoir there are 2 quarts. Oil moves from the engine to the reservoir and back at a rate of 1 quart/minute, and the filter removes a fraction of the dirt from the oil returning from the reservoir to the engine.



a. [4 points] Suppose that x_1 and x_2 are measured in grams. A model for the amount of dirt in either compartment is

$$x_1' = -\frac{1}{3}\,x_1 + \frac{3}{5}\left(\frac{1}{2}\right)x_2 + 3, \quad x_2' = \frac{1}{3}\,x_1 - \frac{1}{2}\,x_2.$$

How much dirt is added to the oil in the engine? Why is there the term $\frac{1}{3}\,x_1$ in each equation, and why does it have this form? How much of the dirt in the oil is removed by the filter, and how do you know?

Solution: The addition term is the $+3$ in the equation for x_1', as that's the one place we have a constant addition of dirt. Thus we're adding an impressive 3 g/min of dirt.

The term $\frac{1}{3}x_1 = (1 \text{ quart/min})(\frac{x_1}{3}\text{g/quart})$ is the rate at which dirt is moved from the engine (at 1 quart/min, with a concentration of $\frac{x_1}{3}$ g/quart) to the reservoir.

Finally, the filter removes 40% of the dirt: the term $\frac{1}{2}x_2$ in the last equation represents the dirt removed from the reservoir, of which $\frac{3}{5} = 60\%$ arrives at the engine.

b. [4 points] Find the equilibrium solution(s) for this system. What is the physical meaning of the equilibrium solution?

Solution: Equilibrium solutions are constant, so we have

$$0 = -\frac{1}{3}x_1 + \frac{3}{10}x_2 + 3, \quad 0 = \frac{1}{3}x_1 - \frac{1}{2}x_2.$$

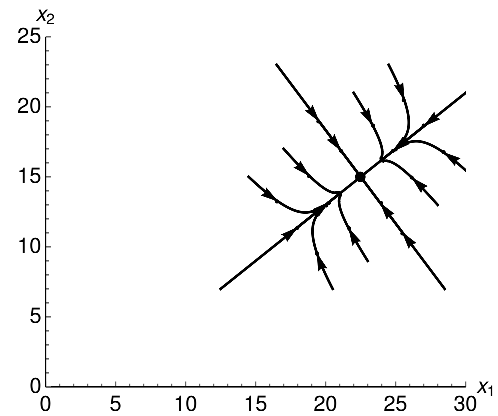
Adding the equations, we have $0 = -\frac{2}{10}x_2 + 3$, so that $x_2 = 15$. Then $x_1 = \frac{3}{2}x_2 = \frac{45}{2} = 22.5$. These are the long-term amounts of dirt that we expect to find in the engine and reservoir.

Problem 5, continued. We are considering the system

$$x_1' = -\frac{1}{3}\,x_1 + \frac{3}{5}\left(\frac{1}{2}\right) x_2 + 3, \quad x_2' = \frac{1}{3}\,x_1 - \frac{1}{2}\,x_2.$$

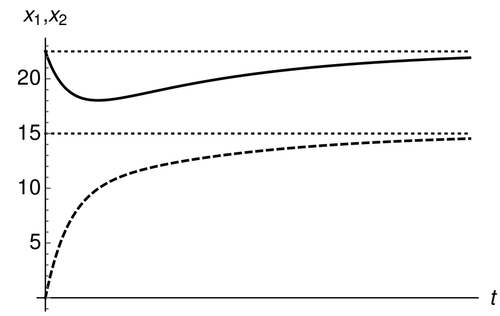
c. [4 points] The eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} -1/3 & 3/10 \\ 1/3 & -1/2 \end{pmatrix}$ are, approximately, $\lambda_1 = -0.75$ and $\lambda_2 = -0.1$, with $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$. Sketch a phase portrait for this system.

Solution: Note that we expect the phase portrait for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, shifted to the equilibrium point $(22.5, 15)$. The phase portrait is a stable node, as shown in the figure below.



d. [4 points] Suppose that, somehow, we start with the initial condition $x_1(0) = 22.5$ and $x_2(0) = 0$. Use your work in (b) to sketch, approximately, what you expect x_1 and x_2 to look like as functions of time.

Solution: From the phase portrait, above, we see that x_2 will increase, asymptotically approaching $x_2 = 15$, and x_1 will initially decrease slightly, then increase to $x_1 = 22.5$. This gives the component curves shown below, with x_1 solid and x_2 dashed, and the equlibria shown dotted.



6. [15 points] In lab 1 we considered the Gompertz equation, $y' = r\,y\,\ln(K/y)$. We explore this further in this problem.

a. [5 points] Consider the initial condition $y(0) = 1$. Find a linear approximation to the Gompertz equation that is valid near this initial condition. Under what conditions would you expect your approximation to be accurate?

Solution: Noting that $y' = r\,y(\ln(K) - \ln(y))$, we can use the Taylor expansion of $\ln(y) = 0 + (y-1) + \cdots$ to linearize the equation. To retain only linear terms, we truncate the log at the constant term 0, and so have $y' = r\ln(K)\,y$. This is a reasonable approximation, though a little bit of a fudge as we haven't expanded the y term in the equation.

If we are slightly more careful, we also expand the linear term y as $y = 1 + (y-1)$. Then $y' = r\,(1 + (y-1))(\ln(K) - (0 + (y-1)))$. To retain only terms in $(y-1)^0$ or $(y-1)^1$ we must truncate the expansion of the logarithm at the constant (0) term, so that $y' = r\,(1 + (y-1))(\ln(K)) = r\ln(K)\,y$.

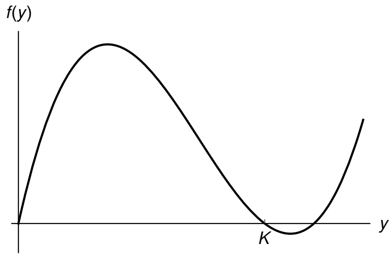
Alternately, if we expand the expression and then retain only linear terms in y, we obtain $y' = r\ln(K) + r\ln(K)(y-1) - r(1)(y-1) = r(\ln(K) - 1)y + r$. This retains a constant term r because $y = 1$ isn't a critical point of the equation.

In any case, this is valid when y is near 1, and as y moves away from that we would expect the approximation to rapidly get worse.

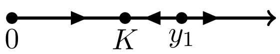
b. [5 points] We found that for y near K, the Gompertz equation is approximated as $y' = -rK(y-K)$. Solve this and explain what its solution tells us about solutions to the Gompertz equation.

Solution: We solve by separation: $y'/(y-K) = -rK$, so that $\ln|y-K| = -rKt + C'$. Exponentiating both sides, and letting $C = \pm e^{C'}$, we have $y = K + Ce^{-rKt}$. This says that if we start with an initial condition near $y = K$, we expect the solution to converge to $y = K$: that is, the equilibrium solution $y = K$ is asymptotically stable.

c. [5 points] If we retain two terms from the Taylor expansion of $\ln(K/y)$ near $y = K$, we obtain the cubic differential equation $y' = f(y)$, where $f(y)$ is shown in the figure to the right. Sketch a phase line for this equation and explain what it suggests about the long-term behavior of the tumor.



Solution: Designating the critical point above $y = K$ as $y = y_1$, we have the phase line shown below.



This suggests that for any initial condition $y(0) = y_0$ with $0 < y_0 < y_1$, the solution will converge to $y = K$ as $t \to \infty$. However, if $y_0 > y_1$, the solution becomes unbounded as $t \to \infty$.

7. [12 points] For each of the following, give an example as indicated, and a short (one or two sentence) explanation for how your example satisfies the indicated criteria.

a. [4 points] Give an example of an autonomous first-order differential equation with two equilibrium solutions, neither of which are stable.

Solution: An example is $y' = y^2(1-y)^2$. It has the two equilibrium solutions $y = 0$ and $y = 1$, but for all $y \neq 0, 1$, $y' > 0$. Thus neither is stable.

b. [4 points] Give an example of a linear, constant-coefficient system of two differential equations whose phase portrait is a stable counterclockwise spiral.

Solution: An example is $\mathbf{x}' = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \mathbf{x}$. We see that the eigenvalues are $\lambda = -1 \pm 2i$, so we have a stable spiral. To check that it is counterclockwise, we consider the direction of the trajectory at $(1, 0)$, where $\mathbf{x}' = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, that is, to the left and up, giving a counterclockwise spiral.

c. [4 points] Give an example of a linear, constant-coefficient system of two differential equations that has a critical point that is not at the origin.

Solution: All we need is that the system have a forcing, e.g., $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. The critical point of this is at $(1, -1)$.