Math 216 — Second Midterm

27 March, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **DO NOT use Laplace transforms**.

a. [7 points] Find the general solution to $2U''(t) + 12U'(t) + 16U(t) = 12e^{4t}$.

Solution: This problem is nonhomogeneous, linear, and constant-coefficient. The general solution will be $U = U_c + U_p$, where U_c is the general solution to the complementary homogeneous problem. For this we look for a solution $U = e^{\lambda t}$. Plugging in to the homogeneous equation, we have $2\lambda^2 + 12\lambda + 16 = 2(\lambda + 4)(\lambda + 2) = 0$. Thus $\lambda = -4$ or $\lambda = -2$, so that U_c is given by $U_c = c_1 e^{-4t} + c_2 e^{-2t}$.
Then, to find Up, we use undertermined coefficients and look for a solution of the form $U_p = Ae^{4t}$. Plugging into the differential equation, we have $A(2(16) + 12(4) + 16)e^{4t} = A(96)e^{4t} = 12e^{4t}$, so that $A = \frac{12}{96} = \frac{1}{8}$, and

$$U = c_1 e^{-4t} + c_2 e^{-2t} + \frac{1}{8} e^{4t}.$$

b. [8 points] Find the solution to the initial value problem $y''(t) + 6y'(t) + 9y(t) = 3e^{-3t}$, $y(0) = 0$, $y'(0) = 1$.

Solution: Again, the general solution will be $y = y_c + y_p$. For y_c, the characteristic equation is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$, so $\lambda = -3$, twice, and $y_c = c_1 e^{-3t} + c_2 te^{-3t}$.
For y_p, we would use the method of undetermined coefficients and guess $y_p = ae^{-3t}$; however, this is part of y_c, so we must multiply by t^2 in order for that not to be the case. Thus we guess $y_p = at^2 e^{-3t}$. Then $y_p' = (-3at^2 + 2at)e^{-3t}$, and $y_p'' = (9at^2 - 6at + 2a)e^{-3t}$, so that, plugging in, we have

$$\begin{aligned} (9at^2 - 6at + 2a)e^{-3t} + 6(-3at^2 + 2at)e^{-3t}) + 9at^2)e^{-3t} &= 3e^{-3t} \\ 2ae^{-3t} &= 3e^{-3t}, \end{aligned}$$

and $y_p = \frac{3}{2} t^2 e^{-3t}$. The general solution is therefore $y = c_1 e^{-3t} + c_2 te^{-3t} + \frac{3}{2} t^2 e^{-3t}$.
Applying the initial conditions, we have $y(0) = c_1 = 0$ and $y'(0) = c_2 = 1$. Thus $y = te^{-3t} + \frac{3}{2} t^2 e^{-3t}$.

2. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **DO use Laplace transforms**.

a. [7 points] Find the solution to the initial value problem $y'' + 4y' + 20y = 0$, $y(0) = 1$, $y'(0) = 5$.

Solution: Taking the Laplace transform of both sides of the equation, we have $\mathcal{L}\{y'' + 4y' + 20y\} = 0$, so that, with $Y = \mathcal{L}\{y\}$,

$$s^2Y - s - 5 + 4(sY - 1) + 20Y = 0,$$

so that

$$Y = \frac{s+9}{s^2+4s+20} = \frac{(s+2)+7}{(s+2)^2+16}.$$

Taking the inverse transform, we have

$$y = \mathcal{L}^{-1}\{\frac{s+2}{(s+2)^2+16}\} + \mathcal{L}^{-1}\{\frac{7}{(s+2)^2+16}\} = e^{-2t}\cos(4t) + \frac{7}{4}e^{-2t}\sin(4t).$$

b. [8 points] Find the solution to the initial value problem $y'' + 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 0$.

Solution: Proceding as above, the forward transform gives

$$s^2Y + 3sY + 2Y = \frac{1}{s+1},$$

so that

$$Y = \frac{1}{(s+1)^2(s+2)}.$$

To find the inverse transform, we use partial fractions, letting

$$\frac{1}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}.$$

Clearing the denominator,

$$1 = A(s+1)(s+2) + B(s+2) + C(s+1)^2.$$

Taking $s = -1$ and $s = -2$, we find $B = 1$ and $C = 1$. Then, if $s = 0$, $1 = 2A + 3$, and $A = -1$.

$$y = \mathcal{L}^{-1}\{-\frac{1}{(s+1)}\} + \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\} + \mathcal{L}^{-1}\{\frac{1}{(s+2)}\} = -e^{-t} + te^{-t} + e^{-2t}.$$

3. [14 points] Suppose that $L[y] = y'' + p(t)y' + q(t)y$. *(Note that $L[y]$ here is a differential operator, not the Laplace transform $\mathcal{L}\{y\}$.)*

a. [7 points] If $L[t^2] = 2 + 2tp(t) + t^2q(t) = 0$ and $L[t^2\ln(t)] = (2\ln(t) + 3) + (2t\ln(t) + t)p(t) + t^2\ln(t)q(t) = 0$, which, if any, of the following functions y are solutions to $L[y] = 0$ on the domain $t > 0$? Which, if any, give a general solution on this domain? Why? (In these expressions, c_1 and c_2 are real constants.)

$y_1 = 5t^2$ $\quad y_2 = 5t^2(1 + 2\ln(t))$ $\quad y_3 = c_1t^2 + c_2t^2\ln(t)$

$y_4 = -t^2\ln(t)$ $\quad y_5 = t^4\ln(t)$ $\quad y_6 = c_1t^2(1 + \ln(t))$

$y_7 = t^{-2}\ln(t)$ $\quad y_8 = W[t^2, t^2\ln(t)] = t^3$ $\quad y_9 = c_1(5t^2 - 2c_2\ln(t))$

Solution: Note that the Wronskian $W[t^2, t^2\ln(t)]$ is correctly given in y_8: $W[t^2, t^2\ln(t)] = t^2(2t\ln(t) + t) - t^3\ln(t) = t^3$. Thus these two functions are linearly independent, and a general linear combination of the two will give the general solution. Further, because L is linear, any linear combination of the two will be a solution. Thus all of y_1, y_2, y_3, y_4, and y_6 are solutions. Those solutions with an arbitrary constant multiplying each of linearly independent solutions are general solutions; this is only y_3.

b. [7 points] Now suppose that $p(t) = 2$ and $q(t) = 10$, and let $L[y] = y'' + 2y' + 10y = g(t)$. For what $g(t)$ will the steady state solution to this problem be constant? Solve your equation with this $g(t)$ and explain how your solution confirms that your $g(t)$ is correct.

Solution: We will get a constant steady state whenever $g(t) = k \in \mathbb{R}$. Note that in this case the characteristic equation is $\lambda^2 + 2\lambda + 10 = (\lambda + 1)^2 + 9 = 0$, so $\lambda = -1 \pm 3i$, and $y_c = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t)$. The particular solution is $y_p = k/10$. Thus $y = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t) + \frac{k}{10}$, and as $t \to \infty$ we see that $y_c \to 0$ and $y \to \frac{k}{10}$, a constant.

4. [15 points] In lab 2 we considered the van der Pol oscillator, modeled by the equation, $x'' + \mu(x^2 - 1)x' + x = 0$. Recall that there is a single critical point for this system, $x = 0$, near which we may model the behavior of the oscillator with the linear equation $x'' - \mu x' + x = 0$.

a. [5 points] Suppose that $\mu = -1$. Find the amplitude of the solution to the linear problem with initial condition $x(0) = 2$, $x'(0) = 3$. What is the time after which this amplitude never exceeds some value a_0?

Solution: If $\mu = -1$, the characteristic equation for the problem is $\lambda^2 + \lambda + 1 = 0$, so $\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The general solution to the problem is therefore $x = c_1 e^{-t/2}\cos(\frac{\sqrt{3}}{2}\,t) + c_2 e^{-t/2}\sin(\frac{\sqrt{3}}{2}\,t)$. Applying the initial condition, $x(0) = c_1 = 2$, and $x'(0) = -1 + \frac{\sqrt{3}}{2}\,c_2 = 3$, so $c_2 = 8/\sqrt{3}$. The solution to the initial value problem is $x = 2e^{-t/2}\cos(\sqrt{3}\,t) + (8/\sqrt{3})e^{-t/2}\sin(\sqrt{3}\,t)$.

The amplitude of this solution is $R = \sqrt{4 + 64/3}\,e^{-t/2} = \sqrt{76/3}\,e^{-t/2}$, so this is less than a_0 when $e^{-t/2} < a_0/\sqrt{76/3}$, or, when $t > 2\ln(\sqrt{76/3}/a_0)$.

b. [5 points] Suppose we force the linear system with an oscillatory input, so that we are considering $x'' - \mu x' + x = \cos(\omega t)$ (and $\omega \neq 0$). For what values of μ will the system have an oscillatory steady-state solution with frequency ω?

Solution: Note that the characteristic equation for this problem is $\lambda^2 - \mu\lambda + 1 = 0$, so that $\lambda = \frac{1}{2}\mu \pm \frac{1}{2}\sqrt{\mu^2 - 4}$. Thus if $\mu > 0$ we are guaranteed at least one positive root, and the homogeneous solution will not decay. If $\mu = 0$, we have no damping and the solution to the problem will either be $y = c_1\cos(2t) + c_2\sin(2t) + R\cos(\omega t - \delta)$ or, if $\omega = 2$, a growing solution. In either case we do not have an oscillatory steady-state solution with frequency ω.

If $-2 < \mu < 0$, the homogeneous solution will be a decaying oscillatory solution, and if $\mu \leq -2$ it will be a decaying exponential (and te^{-2t} if $\mu = -2$). In either case the steady-state solution will be $y_p = a\cos(\omega t) + b\sin(\omega t)$, and hence be oscillatory with frequency ω.

Thus, we need $\mu < 0$.

c. [5 points] Suppose that, for some choice of μ, the system $x'' - \mu x' + x = \cos(\omega t)$ has an oscillatory steady-state solution, and that the gain function $G(\omega)$ for this solution is shown to the right, below. If the steady-state solution to the problem is $y_{ss} = R\cos(t - \pi/2)$, what are the R in the solution, and ω in the forcing term? Why?

Solution: The steady state solution to the problem has frequency ω, so $\omega = 1$. Then, if $\omega = 1$, we see from the gain function that $R = 1$.

5. [14 points] In each of the following we consider a linear, second order, constant coefficient operator L, so that $L[y] = 0$ is a homogenous differential equation. (Note, however, that the operator L may be different in each of the parts below.) Let $y(0) = y_0$ and $y'(0) = v_0$, where y_0 and v_0 are real numbers.

a. [7 points] If the general solution to the equation $L[y] = 0$ is $y = c_1e^{-3t}\cos(2t) + c_2e^{-3t}\sin(2t)$, what is the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$?

Solution: The general solution tells us that the two roots of the characteristic polynomial are $\lambda = -3 \pm 2i$. Thus the characteristic polynomial is $(\lambda + 3)^2 + 4 = \lambda^2 + 6\lambda + 13$, and so $L[y] = y'' + 6y' + 13y$. The forward transform of this will be $\mathcal{L}\{y'' + 6y' + 13y\} = -sy_0 - v_0 + s^2Y - 6y_0 + 6sY + 13Y = 0$, and

$$Y = \frac{sy_0 + v_0 + 6y_0}{s^2 + 6s + 13}.$$

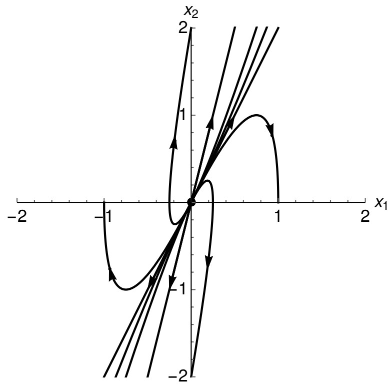
b. [7 points] Now suppose that we are solving $L[y] = k$, for some constant k, and that y_0 and v_0 are both zero (so that $y(0) = y'(0) = 0$). If $Y(s) = \mathcal{L}\{y(t)\}$ is $Y = \frac{k}{s(s+3)(s+4)}$, what is the differential equation we are solving, and the general solution to the complementary homogeneous problem? Explain how you know your answer is correct.

Solution: Note that $\mathcal{L}\{k\} = \frac{k}{s}$. Therefore the operator L is $L = (D + 3)(D + 4)$, to get the quadratic term multiplying s in the denominator of Y. That is, $L = D^2 + 7D + 12$, so that $L[y] = y'' + 7y' + 12y$, and the general solution to the problem is $y = c_1e^{-3t} + c_2e^{-4t}$.

6. [15 points] Each of the following concerns a linear, second order, constant coefficient differential equation $y'' + py' + qy = 0$.

a. [7 points] If the general solution to the problem is $y = c_1e^{2t} + c_2e^{4t}$, sketch a phase portrait for the system.

Solution: In matrix form, the general solution to the system will be $\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$. Thus eigenvectors for the system will be $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, with $\lambda = 2$ and $\lambda = 4$, respectively. We therefore have the phase portrait shown.



b. [8 points] Now suppose that for some real-valued α, we have $p = 2\alpha$ and $q = 1$, so that we are considering $y'' + 2\alpha y' + y = 0$. For what values of α, if any

(i) do all solutions to the differential equation decay to zero?

(ii) are there solutions that do not decay to zero?

(iii) will the general solution be a decaying sinusoidal function?

Solution: We see that the characteristic polynomial is in this case $\lambda^2 + 2\alpha\lambda + 1 = 0$, so that $\lambda = -\alpha \pm \sqrt{\alpha^2 - 1}$. Thus: **(i)** The square root can never be larger in magnitude than α, so for all $\alpha > 0$ we will have solutions that decay to zero. **(ii)** If $\alpha \leq 0$, there will be solutions that do not decay to zero, and, in fact, if $\alpha < 0$ all solutions will diverge to infinity. **(iii)** If $|\alpha| < 1$, the square root will be imaginary, so that $\lambda = -\alpha \pm i\sqrt{1 - \alpha^2}$. In this case solutions if $0 < \alpha < 1$ we will have decaying sinusoidal solutions.

7. [12 points] For each of the following give an example, as indicated, and provide a short (one or two sentence) explanation of why your answer is correct.

a. [4 points] Give an example of an initial value problem with a linear, second-order, homogeneous differential equation for which there is no guarantee of a unique solution.

Solution: A linear, second-order, homogeneous differential equation has the form $y'' + p(t)y' + q(t)y = 0$. For there to be no guarantee of a unique solution one or both of the functions $p(t)$ and $q(t)$ must be discontinuous at the initial condition. Thus, one such example is $y'' + t^{-1}y' + y = 0$, $y(0) = y_0$, $y'(0) = v_0$.

b. [4 points] Give an example of a linear, second-order, constant-coefficient, nonhomogeneous differential for which we cannot use the method of undetermined coefficients. What form will the general solution to your equation take?

Solution: We know that undetermined coefficients fails when the forcing is not polynomial, sinusoidal, exponential or a product of these. Thus one such example is $y''+2y'+y = t^{-1}$. The solution will be of the form $y = c_1e^{-t} + c_2te^{-t} + u_1(t)e^{-t} + u_2(t)te^{-t}$, using variation of parameters.

c. [4 points] Give an example of a linear, second-order, nonhomogeneous differential equation for which the Laplace transform of the dependent variable y could be $\mathcal{L}\{y(t)\} = Y(s) = \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3}$.

Solution: The equation is second-order, so the general solution will be of the form $y = c_1y_1 + c_2y_2 + y_p$. Here we see that the solution is $y = e^{-t} + e^{-2t} + e^{-3t}$. Recognizing this, there are a number of ways to proceed. The first is to identify two of the three exponentials must be part of the complementary homogeneous problem, so that the characteristic polynomial is one of $(\lambda+1)(\lambda+2) = \lambda^2+3\lambda+2$, $(\lambda+2)(\lambda+3) = \lambda^2+5\lambda+6$, or $(\lambda+1)(\lambda+3) = \lambda^2+4\lambda+3$. With these, the equation must be one of $y''+3y'+2y = 2e^{-3t}$, $y'' + 5y' + 6y = 2e^{-t}$, or $y'' + 4y' + 3y = -e^{-2t}$.

A second approach is to take this solution and create a differential equation; for example, if $y = e^{-t} + e^{-2t} + e^{-3t}$, then $y'' = e^{-t} + 4e^{-2t} + 9e^{-3t}$ is a linear, second-order nonhomogeneous differential equation for which the Laplace transform could be as indicated. With some care, it is possible to pick other solutions as well.