Math 216 — Final Exam

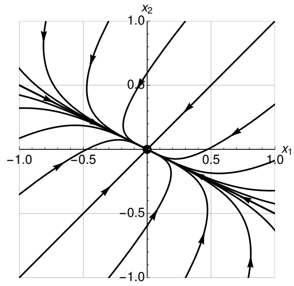
26 April, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] Six matrices and their eigenvalues and eigenvectors are given below. Use this information to answer the questions below. Be sure that you explain your answers.

$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{A}_5$	$\mathbf{A}_6$
$\begin{pmatrix}-1 & 2\\ -1 & -3\end{pmatrix}$	$\begin{pmatrix}2 & 2\\ 1 & 3\end{pmatrix}$	$\begin{pmatrix}-2 & 2\\ 1 & -3\end{pmatrix}$	$\begin{pmatrix}-1 & 3\\ 2 & -2\end{pmatrix}$	$\begin{pmatrix}-2 & -2\\ -1 & -3\end{pmatrix}$	$\begin{pmatrix}-3 & -1\\ 1 & -1\end{pmatrix}$
$\lambda_{1,2} = -2 \pm i$	$\lambda_{1,2} = 1, 4$	$\lambda_{1,2} = -4, -1$	$\lambda_{1,2} = -4, 1$	$\lambda_{1,2} = -4, -1$	$\lambda_{1,2} = -2, -2$
$\mathbf{v}_1 = \begin{pmatrix}2\\ -1+i\end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix}-2\\ 1\end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix}-1\\ 1\end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix}-1\\ 1\end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix}1\\ 1\end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix}-1\\ 1\end{pmatrix}$
$\mathbf{v}_2 = \begin{pmatrix}2\\ -1-i\end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix}1\\ 1\end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix}2\\ 1\end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix}3\\ 2\end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix}-2\\ 1\end{pmatrix}$	$\mathbf{w} = \begin{pmatrix}1\\ 0\end{pmatrix}$

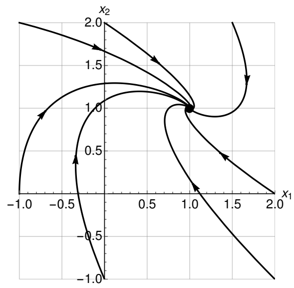
a. [6 points] Write a linear system involving one of the $\mathbf{A}_j$ that could have the phase portrait shown to the right.



Solution: We note that there is a critical point at the origin, so we are solving a homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. All solutions approach the origin, and there are two straight-line solutions, so there must be two real negative eigenvalues. This means our coefficient matrix is either $\mathbf{A}_3$ or $\mathbf{A}_5$. Then we note that the eigenvalue associated with an eigenvector with negative slope is larger, because all trajectories not on an eigenvector asymptotically approach a line in the direction of that eigenvector. This is the case for $\mathbf{A}_5$, where $\lambda = -4$ and $\lambda = -1$, and the eigenvector associated with $\lambda = -1$ has negative slope—and the eigenvectors correspond to the lines $y = x$ and $y = x/2$ seen in the phase portrait. Thus our system is

$$\mathbf{x}' = \begin{pmatrix}-2 & -2\\ -1 & -3\end{pmatrix}\mathbf{x}.$$

b. [6 points] Write a linear system involving one of the $\mathbf{A}_j$ that could have the phase portrait shown to the right.



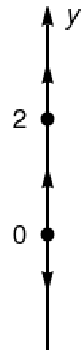
Solution: Note that the critical point is in this case at $(1, 1)$, not the origin. Thus our system must be non-homogeneous, $\mathbf{x}' = \mathbf{A} + \mathbf{k}$. There are no straight-line trajectories shown in the phase portrait, so the eigenvalues of $\mathbf{A}$ must be complex, which is the case for $\mathbf{A}_1$; further, $\mathbf{A}_1$ predicts that above the critical point slopes will be $\begin{pmatrix}x_1'\\ x_2'\end{pmatrix} = \mathbf{A}_1\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}2\\ -3\end{pmatrix}$, which is consistent with the phase portrait. Finally, for the critical point to be at $(1, 1)$, we must have $\mathbf{0} = \begin{pmatrix}-1 & 2\\ -1 & -3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} + \mathbf{k}$, so that $\mathbf{k} = \begin{pmatrix}-1\\ 4\end{pmatrix}$, and our system is

$$\mathbf{x}' = \begin{pmatrix}-1 & 2\\ -1 & -3\end{pmatrix}\mathbf{x} + \begin{pmatrix}-1\\ 4\end{pmatrix}.$$

2. [12 points] For each of the following, give an example, as indicated.

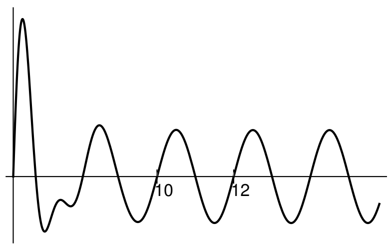
a. [4 points] Give a first-order differential equation that could have the phase line shown to the right.

Solution: We see that there are critical points at $y = 0$ and $y = 2$, and that the first is unstable and the second semistable. This suggests the equation $y' = y(y-2)^2$. Checking, we see that for $y < 0$, $y' < 0$, and $y > 0$ gives $y' > 0$ (for $y \neq 2$), so this works.



b. [4 points] Give a second-order, linear, constant-coefficient, nonhomogeneous differential equation that could have the response shown to the right.

Solution: We note this has a likely decaying oscillatory transient, and a steady sinusoidal response. Thus we expect the equation is something like $y'' + y' + 2y = \cos(\omega t)$. We note that the period of the steady response is 2, so $\omega = \pi$.



c. [4 points] Give a system of two linear, first-order, constant-coefficient differential equations which have an isolated critical point at the origin that is an unstable saddle point.

Solution: A system such as $\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \mathbf{x}$ will have the desired characteristics. Eigenvalues are $\lambda = \pm 1$, and because $\det(\mathbf{A}) \neq 0$ the origin is the only critical point and is thus isolated.

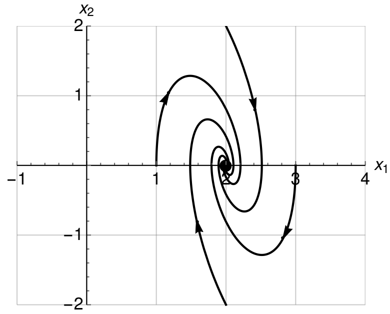
3. [12 points] Suppose a model for a physical system (e.g., a circuit or a mass-spring system) is given by the differential equation $L[y] = y'' + ay' + by = k$ (where a, b, and k are real numbers).

a. [4 points] If the solution to the problem with some initial conditions is $y = e^{-t}\cos(2t) - e^{-t}\sin(2t) + 2$, what can you say about a, b, and k?

Solution: We see that eigenvalues are $\lambda = -1 \pm 2i$, so the characteristic polynomial is $p(\lambda) = (\lambda+1)^2 + 4 = \lambda^2 + 2\lambda + 5$ and $a = 2$, $b = 5$. Then $y_p = 2$, so $k = 10$.

b. [4 points] If the solution to the problem with some initial conditions is $y = e^{-t}\cos(2t) - e^{-t}\sin(2t) + 2$, sketch a phase portrait for the system. Be sure it is clear how you obtain your solution.

Solution: This is a spiral sink centered on $(2, 0)$. Note that $y(0) = 3$, and as t increases y looks like $(\cos(2t), -\sin(2t))$, so the spiral is clockwise. This can also be seen by writing the equation as a system and checking the direction of motion.



c. [4 points] Now suppose that the solution to the problem with some initial conditions is $y = e^{-t}\cos(2t) - e^{-t}\sin(2t) + 2$, and that at some time $t = t_0$ we remove the forcing term (k). Write a single differential equation you could solve to find y for all $t \geq 0$. What initial conditions apply at $t = 0$?

Solution: The equation is $y'' + 2y' + 5y = 10(1 - u_{t_0}(t))$; we can see the initial conditions from the solution: $y(0) = 3$, $y'(0) = -3$.

4. [12 points] Consider the predator-prey model with harvesting (harvesting here implies hunting by humans, e.g., fishing if the populations are fish) given by

$$x' = x\,(3 - x - y) - 2, \quad y' = y\,(-3 + x)\,.$$

Note that as x and y are populations, we must have $x, y \geq 0$.

a. [3 points] Explain what each term in the equation for x models. Is x or y the predator? Which population is being harvested?

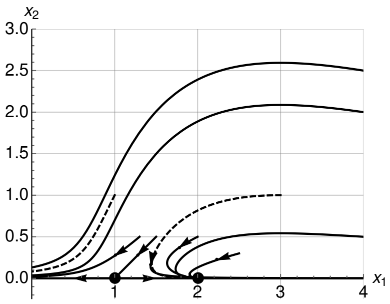
Solution: In the equation for x, the term $3x$ is a birth/death rate term; $-x^2$ is a logistic resource limitation term; $-xy$ is the species interaction term, and as it is negative we know x must be the prey; and the -2 must be the harvesting term, so x is also being harvested.

b. [7 points] By doing an appropriate linear analysis, sketch a phase portrait for this system.

Solution: First we find critical points: if $y' = 0$, we need $y = 0$ or $x = 3$. Then, if $x' = 0$ and $x = 3$, we have $-3y - 2 = 0$, so that $y = -\frac{2}{3}$. This doesn't make sense for our model, so we ignore it. If $y = 0$, $x' = 0$ requires $-x^2 + 3x - 2 = -(x-2)(x-1) = 0$, so $x = 1$ or $x = 2$. The physically relevant critical points are therefore $(1, 0)$ and $(2, 0)$.

Then, the Jacobian for the system is $\mathbf{J} = \begin{pmatrix} 3 - 2x - y & -x \\ y & -3 + x \end{pmatrix}$.

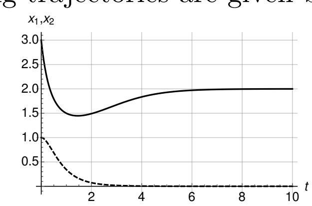
At $(1, 0)$, $\mathbf{J}(1, 0) = \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}$, so that eigenvalues of the linearized system are $\lambda = 1, -2$. Corresponding eigenvectors are $\mathbf{v} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $\mathbf{v} = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$. At $(2, 0)$, $\mathbf{J}(2, 0) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}$, so $\lambda = -1$ with $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We could do the analysis without the generalized eigenvector, but it satisfies $\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so that $\mathbf{w} = \begin{pmatrix} 0 & -\frac{1}{2} \end{pmatrix}^T$. Note that this means that a trajectory starting immediately below the critical point will initially move to the right.

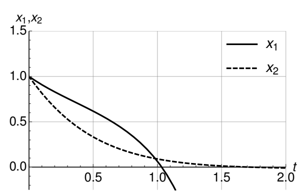


Finally, note that the y-nullclines ($y' = 0$) are $y = 0$ and $x = -3$; the x-nullcline is harder to visualize, but is given by $y = 3 - x - \frac{2}{x}$. Putting these together, we get the phase portrait shown to the right, above.

c. [2 points] Based on your answer to (b), sketch what you expect the behavior of the solution to the system will be as a function of time if $x(0) = 3$ and $y(0) = 1$. How would you expect this to differ from the behavior with the initial condition $x(0) = 1$, $y(0) = 1$?

Solution: Starting at $(3, 1)$ the trajectory should move to the right and down, overshooting the critical point at $(2, 0)$ but then coming back to converge to it. From $(1, 1)$, we move to the left and down until $x = 0$. At that point population x vanishes and $y' = -3y$, so y also decays to zero. These are shown by the two dashed curves above. The corresponding trajectories are given below.





5. [10 points] Consider the linear system

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -1 & 0 & \alpha^2 \\ 0 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

a. [5 points] For what values of α, if any, will all solutions to the system remain bounded as $t \to \infty$?[1]

Solution: Solutions will be unbounded if there are any eigenvalues λ of the coefficient matrix for which $\mathrm{Re}(\lambda) > 0$, so we find the eigenvalues of the matrix. Eigenvalues satisfy $\det(\mathbf{A}-\lambda\mathbf{I}) = (-1-\lambda)((-2-\lambda)(-1-\lambda)-0)-0+\alpha^2(0-(-2-\lambda)) = (-2-\lambda)((\lambda+1)^2-\alpha^2) = 0$. Thus $\lambda = -2$ or $\lambda = -1 \pm \alpha$, and all solutions will remain bounded provided $|\alpha| \le 1$ (that is, $-1 \le \alpha \le 1$).

b. [5 points] Now suppose that $\alpha = 2$. Are there any initial conditions for which solutions to the system will remain bounded? If so, what are they? Explain.

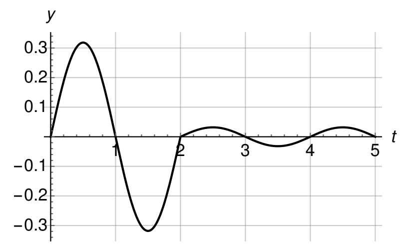
Solution: From (a), the eigenvalues are $\lambda = -2$, $\lambda = -3$, and $\lambda = 1$. Thus we will have bounded solutions if we start with an initial condition that includes of only the solutions associated with the first two eigenvalues. To see what these are, we first find the eigenvectors. For the three eigenvalues we have: for $\lambda = -3$, $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$; for $\lambda = -2$, $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$; and for $\lambda = 1$, $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} -2 & 0 & 4 \\ 0 & -3 & 2 \\ 1 & 0 & -2 \end{pmatrix}$. Thus for $\lambda = -3$, $v_1 = -2v_3$ and $v_2 = -2v_3$, and we can take $\mathbf{v} = \begin{pmatrix} 2 & 2 & -1 \end{pmatrix}^T$. For $\lambda = -2$, we can take $\mathbf{v} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$. Finally, for $\lambda = 1$, we have $v_1 = 2v_3$ and $3v_2 = 2v_3$, and can take $\mathbf{v} = \begin{pmatrix} 6 & 2 & 3 \end{pmatrix}$. The general solution to the problem is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} e^{t}.$$

Any initial condition that is a linear combination of the first two eigenvectors will remain bounded as $t \to \infty$. (This is the plane $x = 2u$, $y = 2u + v$, $z = -u$, for $u, v \in \mathbb{R}$.)

[1] *Possibly useful:* $\det\left(\begin{pmatrix} a & 0 & b \\ 0 & c & d \\ e & 0 & f \end{pmatrix}\right) = acf - bce.$

6. [12 points] In lab we considered an electrical system $y'' + \omega_0^2 y = f(t)$ which produced a response similar to that shown in the figure to the right. In this problem, we will take $\omega_0 = \pi$, and use the initial conditions $y(0) = 0$, $y'(0) = 1$.



a. [6 points] If we pick $f(t) = k\delta(t - t_0)$, what is t_0? Solve the problem with this $f(t)$ to find a value of k that results in a solution that could produce a graph similar to this one. Explain your logic.

Solution: We note that the abrupt change in the solution occurs when $t = 2$, so we must have $t_0 = 2$. Then, taking the Laplace transform of both sides of the equation and letting $Y(s) = \mathcal{L}\{y(t)\}$, we have $s^2Y - 1 + \pi^2 Y = e^{-2s}$, so that

$$Y(s) = \frac{1}{s^2 + \pi^2} + \frac{ke^{-2s}}{s^2 + \pi^2}.$$

Inverting,

$$y(t) = \tfrac{1}{\pi}\sin(\pi t) + \tfrac{k}{\pi}u_2(t)\sin(\pi(t-2)) = \tfrac{1}{\pi}\left(1 + ku_2(t)\right)\sin(\pi t),$$

because the period of $\sin(\pi t)$ is 2. Thus if we pick $k = -0.9$, before $t = 2$ the solution is $y = \frac{1}{\pi}\sin(\pi t)$, which looks like the solution shown; and for $t > 2$ we have $y = \frac{0.1}{\pi}\sin(\pi t)$, an oscillation with the reduced amplitude shown.

b. [6 points] Now suppose $f(t) = kI(t)$, where $I(t)$ is the finite-width impulse we used in lab 4. Let $I(t)$ have height 8 and width $\frac{1}{8} = 0.125$, applied at $t = 2$ (that is, $I(t) = 8$ for $2 < t < 2.125$, and is zero elsewhere). Solve the resulting problem to find y. What should k be to produce a solution similar to that graphed above (your work in lab may be useful here)? If you used this value of k in (a), how would the response graph be different?

Solution: Note that $I(t) = 8(u_2(t) - u_{2.125}(t))$. From lab, we expect that picking $k = -1$ will produce a result similar to that shown, as it will partially cancel the initial signal. Then we have the forward transform $s^2Y - 1 + \pi^2 Y = -\frac{8}{s}(e^{-2s} - e^{-2.125s})$, so that

$$Y = \frac{1}{s^2 + \pi^2} - \frac{8(e^{-2s} - e^{-2.125s})}{s(s^2 + \pi^2)}.$$

To invert the second term, we use partial fractions: $\frac{1}{s(s^2+\pi^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+\pi^2}$, so $1 = A(s^2+\pi^2) + (Bs^2 + Cs)$. With $s = 0$, $A = \frac{1}{\pi^2}$, and matching powers of s^2, $B = -A = -\frac{1}{\pi^2}$. The only term in s is Cs, so $C = 0$. Thus, the inverse transform is

$$\begin{aligned} y(t) = \frac{1}{\pi}\sin(\pi t) &- \frac{8}{\pi^2}(u_2(t) - u_{2.125}(t)) \\ &+ \frac{8}{\pi^2}\left(\cos(\pi(t-2))u_2(t) - \cos(\pi(t - 2.125))u_{2.125}(t)\right). \end{aligned}$$

It is not obvious without resorting to more extensive trigonometry than we choose to do here that this is the response seen in the figure, but it can be shown that we do get a similar response (with an amplitude $R \approx 0.062$). If we used this k ($= -1$) in (a), we see that the response would cancel exactly to zero at $t = 2$.

7. [15 points] **DO** complete this problem if you have **NOT** completed the mastery assessment. **DO NOT** complete it if you have completed the mastery assessment. Find explicit, real-valued solutions for each of the following.

a. [7 points] Find $Q(z)$ if $(z+1)Q' = 3Q^2$, $Q(0) = 4$.

Solution: This is nonlinear but separable. Separating, we have $Q^{-2}Q' = 3(z+1)^{-1}$, so that, integrating both sides with respect to z, we have $-Q^{-1} = 3\ln|z+1| + C$, or $Q = -1/(3\ln|z+1| + C)$. The initial condition requires $Q(0) = -1/C = 4$, so that $C = -\frac{1}{4}$, and

$$Q = \frac{-1}{3\ln|z+1| - \frac{1}{4}} = \frac{4}{1 - 12\ln|z+1|}.$$

b. [8 points] Find the general solution to the first-order system $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 13 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Solution: Looking for $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{v}e^{\lambda t}$, we have $\det(\begin{pmatrix} -\lambda & 13 \\ -1 & -6-\lambda \end{pmatrix}) = \lambda^2 + 6\lambda + 13 = (\lambda+3)^2 + 4 = 0$, so $\lambda = -3 \pm 2i$. With $\lambda = -3 + 2i$, we need $\mathbf{v}$ to satisfy $\begin{pmatrix} 3-2i & 13 \\ -1 & -3-2i \end{pmatrix}\mathbf{v} = \mathbf{0}$. We can take $\mathbf{v} = \begin{pmatrix} -3-2i \\ 1 \end{pmatrix}$. Separating the real and imaginary parts of the resulting solution $\mathbf{x} = \mathbf{v}e^{\lambda t}$, we have the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} -3\cos(2t) + 2\sin(2t) \\ \cos(2t) \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} -2\cos(2t) - 3\sin(2t) \\ \sin(2t) \end{pmatrix} e^{-3t}.$$

Alternately, if we solve for $\mathbf{v}$ using the first line of the equation it satisfies, we might take $\mathbf{v} = \begin{pmatrix} 13 \\ -3+2i \end{pmatrix}$, so that

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 13\cos(2t) \\ -3\cos(2t) - 2\sin(2t) \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 13\sin(2t) \\ 2\cos(2t) - 3\sin(2t) \end{pmatrix} e^{-3t}.$$

8. [15 points] **DO** complete this problem if you have **NOT** completed the mastery assessment. **DO NOT** complete it if you have completed the mastery assessment. Find explicit, real-valued solutions for each of the following.

a. [7 points] Find $W(t)$ if $W'' - 2W' - 8W = 24t + 54$, $W(0) = 0$, $W'(0) = 0$.

Solution: The general solution to the problem is $W = W_c + W_p$. For the complementary homogenous solution W_c we look for $W_c = e^{\lambda t}$, so that $\lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2) = 0$ and $\lambda = -2$ or $\lambda = 4$. Thus $W_c = c_1 e^{-2t} + c_2 e^{4t}$. For W_p, we use the method of undetermined coefficients and guess $W_p = at + b$. Plugging in, $-2a - 8at - 8b = 24t + 54$, so that $a = -3$ and $b = -6$. The general solution is $W = c_1 e^{-2t} + c_2 e^{4t} - 3t - 6$.

Initial conditions give $W(0) = c_1 + c_2 + 6 = 0$, and $W'(0) = -2c_1 + 4c_2 - 3 = 0$. Taking four times the first and subtracting the second, $6c_1 = -21$, so $c_1 = -\frac{7}{2}$. Taking twice the first plus the second, $6c_2 = 15$, and $c_2 = \frac{5}{2}$. Thus

$$W = -\frac{7}{2}e^{-2t} + \frac{5}{2}e^{4t} - 3t + 6.$$

b. [8 points] Find $p(t)$ if $p'' + 8p' + 16p = 6\delta(t - 2)$, $p(0) = 0$, $p'(0) = 9$.

Solution: Because of the Dirac delta function, it is easiest to use Laplace transforms for this problem. Transforming both sides of the equation and taking $P(s) = \mathcal{L}\{p(t)\}$, we have $s^2P - 9 + 8sP + 16P = 6e^{-2s}$, so that $P = \frac{9}{(s+4)^2} + \frac{6}{(s+4)^2}e^{-2s}$. Noting that $\mathcal{L}^{-1}\{\frac{1}{(s+4)^2}\} = te^{-4t}$, we have

$$p(t) = 9te^{-4t} + 6(t - 2)e^{-4(t-2)}u_2(t).$$