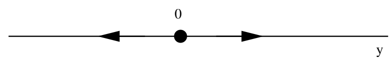
4. [15 points] Consider the differential equation $y' = y\,(y^2 + k)$.

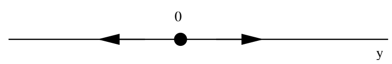
a. [4 points] If $k > 0$, find all equilibrium solutions to this equation. Determine the stability of each and draw a phase diagram.

Solution: Equilibrium solutions are found when $y' = 0$, so we need $y = 0$ or $y^2 = -k$. If $k > 0$ the latter has no (real) solutions, so the only equilibrium solution is $y = 0$. If $y < 0$, the differential equation shows $y' < 0$, and if $y > 0$, $y' > 0$. Thus this is an unstable equilibrium and the phase diagram is as shown below.



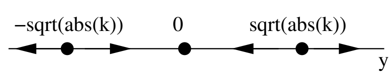
b. [4 points] If $k = 0$, find all equilibrium solutions to this equation. Determine the stability of each and draw a phase diagram.

Solution: In this case the differential equation is $y' = y^3$. Equilibrium solutions are found when $y' = 0$, so the only equilibrium solution is $y = 0$. If $y < 0$, the differential equation shows $y' < 0$, and if $y > 0$, $y' > 0$. Thus this is an unstable equilibrium and the phase diagram is the same as for part (a), as shown below.



c. [4 points] If $k < 0$, find all equilibrium solutions to this equation. Determine the stability of each and draw a phase diagram.

Solution: Finally, if $k < 0$, we have $y' = y(y^2 - |k|)$. Equilibrium solutions are found when $y' = 0$, so we need $y = 0$ or $y^2 = |k|$. Thus our equilibrium solutions are $y = 0$ and $y = \pm\sqrt{|k|}$. We can write the differential equation as $y' = y(y - \sqrt{|k|})(y + \sqrt{|k|})$. Then if $y < -\sqrt{|k|}$ we see that $y' < 0$; if $-\sqrt{|k|} < y < 0$, $y' > 0$; if $0 < y < \sqrt{|k|}$, $y' < 0$, and if $y > \sqrt{|k|}$, $y' > 0$. Thus the two equilibria $y = \pm\sqrt{|k|}$ are unstable and $y = 0$ is stable, as shown in the phase diagram below.



d. [3 points] Use your work from (a)–(c) to draw a bifurcation diagram for this differential equation.

Solution: For the bifurcation diagram we graph y_{eq}, the equilibrium solutions, against the bifurcation paramter, k. This gives the graph shown below.