2. [16 points] Find real-valued solutions to each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [8 points] The general solution to the system $x_1' = 2x_1 + 3x_2$, $x_2' = x_1 + 4x_2$

Solution: This is $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. We know solutions will be of the form $\mathbf{x} = \mathbf{v}\,e^{\lambda t}$, where $\mathbf{v}$ and λ are the eigenvectors and eigenvalues of $\mathbf{A}$. Solving for these, we require

$$\Delta = \det(\mathbf{A} - \lambda\mathbf{I}) = \det(\begin{pmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{pmatrix} = (\lambda-2)(\lambda-4) - 3$$
$$= \lambda^2 - 6\lambda + 5 = (\lambda-5)(\lambda-1) = 0.$$

Thus $\lambda = 1$ or $\lambda = 5$. If $\lambda = 1$, the eigenvector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ satisfies $v_1 + 3v_2 = 0$, so that $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Similarly, if $\lambda = 5$, the eigenvector satisfies $v_1 - v_2 = 0$, and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus the general solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}.$$

b. [8 points] The solution to $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, with $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

Solution: Again, we look for the eigenvalues and eigenvectors of the coefficient matrix. We have $\Delta = (1-\lambda)(-1-\lambda) + 2 = \lambda^2 + 1 = 0$, so that $\lambda = \pm i$. If $\lambda = i$, the eigenvector $\mathbf{v}$ satisfies

$$\begin{pmatrix} 1-i & 2 \\ -1 & -1-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so that from the first row we see we may take $v_1 = 2$ and $v_2 = -1 + i$. Thus a complex-valued solution

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1+i \end{pmatrix} (\cos(t) + i\sin(t)) = \begin{pmatrix} 2\cos(t) + 2i\sin(t) \\ -\cos(t) - \sin(t) + i(\cos(t) - \sin(t)) \end{pmatrix}.$$

The real and imaginary parts of this are linearly independent solutions to the system, and so a real-valued general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2\cos(t) \\ -\cos(t) - \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} 2\sin(t) \\ \cos(t) - \sin(t) \end{pmatrix}.$$

To get the desired initial condition, we take $c_1 = 2$ and $c_2 = 0$, so that

$$\mathbf{x} = \begin{pmatrix} 4\cos(t) \\ -2(\cos(t) + \sin(t)) \end{pmatrix}.$$