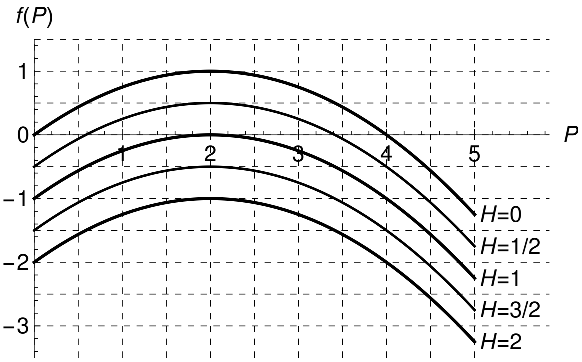
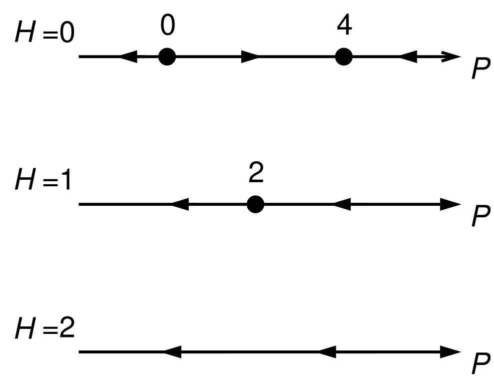
4. [18 points] A model for a population with harvesting (e.g., a population of fish from which fish are caught) is $P' = f(P) = P(1 - \frac{P}{K}) - H$, where K is a limiting population and H the harvesting rate. P and K are measured in some unit—perhaps millions of pounds of fish. Suppose that for some value of K, the graphs of $f(P)$ are as in the graph shown below.

a. [6 points] Plot phase lines for this equation when $H = 0$, $H = 1$ and $H = 2$. For each, identify all equilibrium solutions and their stability.

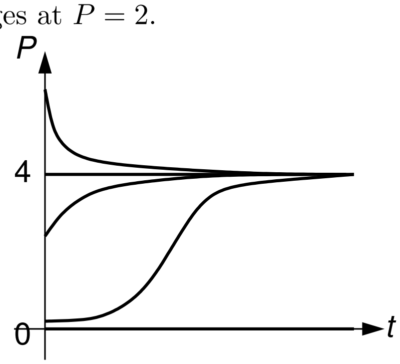


Solution: The phase lines are shown to the right. For $H = 0$, there are two equilibrium solutions, $P = 0$ and $P = 4$, with $P = 4$ being asymptotically stable and $P = 0$ unstable. For $H = 1$, there is a single equilibrium solution, $P = 2$, which is semistable (or, unstable). For $H = 2$ there are no equilibrium solutions, and all initial conditions strictly decrease. (Note that there is some ambiguity in what happens for $P < 0$, which is not shown in the graph and non-physical. Here we sketch the behaviors there by using the equation, which is defined for $P < 0$.)

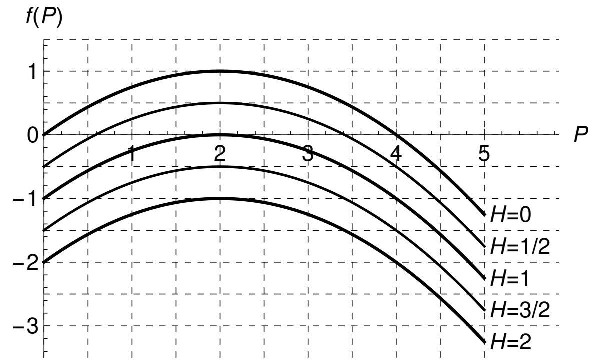


b. [5 points] Sketch qualitatively accurate solution curves for the case $H = 0$. Include enough initial conditions to show all solution behaviors.

Solution: Given the phase line above, we get the curves shown below. Note that the concavity of solutions changes at $P = 2$.

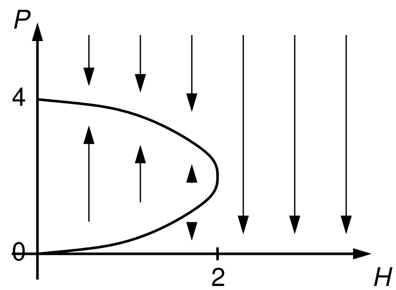


Problem 4, continued. Instructions are reproduced here:
A model for a population with harvesting (e.g., a population of fish from which fish are caught) is $P' = f(P) = P(1 - \frac{P}{K}) - H$, where K is a limiting population and H the harvesting rate. P and K are measured in some unit—perhaps millions of pounds of fish. Suppose that for some value of K, the graphs of $f(P)$ are as in the graph shown below.



c. [4 points] This problem and your work on it provide an example of a model with a bifurcation. Draw the bifurcation diagram for this on the axes provided below.

Solution: The bifurcation diagram is below. The curve shown gives the two branches of a square root, with a base at $H = 2$.



d. [3 points] Explain what your work in the preceding indicates about the long-term survival of the harvested population (fish).

Solution: This indicates that if the harvesting is too high, the population will crash and go to zero. If it is low enough there is a stable larger population and things will continue as one might desire. The transition occurs at $H = 1$; below this, there is a stable "large" population of fish.