1. [15 points] For each of the following, find explicit, real-valued solutions, as indicated.

a. [7 points] The general solution to $x' = 3y$, $y' = 2x + y$

Solution: As a matrix equation, this is $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$. Eigenvalues of $\mathbf{A}$ are given by

$$\det\left(\begin{pmatrix} -\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}\right) = -\lambda(1-\lambda) - 6 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0.$$

Thus $\lambda = -2, 3$. If $\lambda = -2$, the eigenvector satisfies $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Similarly, if $\lambda = 3$, we have $\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The general solution is therefore

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

b. [8 points] The solution to $\mathbf{x}' = \begin{pmatrix} 0 & -2 \\ 2 & -4 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Solution: Eigenvalues of the coefficient matrix satisfy

$$\det\left(\begin{pmatrix} -\lambda & -2 \\ 2 & -4-\lambda \end{pmatrix}\right) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0,$$

so $\lambda = -2$, twice. Eigenvectors satisfy

$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0},$$

so $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. To find a second solution, we look for $\mathbf{x} = (t\mathbf{v} + \mathbf{w})e^{-2t}$, so that

$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{w} = \mathbf{v}.$$

This is $2w_1 - 2w_2 = 1$, so we may take $w_1 = \frac{1}{2}$ and $w_2 = 0$. The general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right) e^{-2t}.$$

Plugging in the initial condition, we have $c_1 + c_2/2 = 3$, and $c_1 = 0$. Thus $c_2 = 6$, and

$$\mathbf{x} = 6 \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right) e^{-2t}.$$