2. [16 points] Let $\mathbf{A}$ be a 2×2 matrix with real entries that has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$ with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a. [6 points] What is the result of each of the following matrix multiplications? Briefly explain your answer for each.

$\mathbf{A}\begin{pmatrix} -1 \\ 1 \end{pmatrix} =$

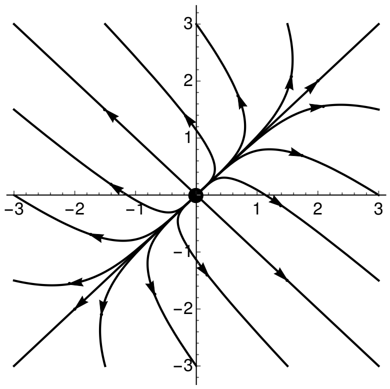
$\mathbf{A}\begin{pmatrix} 2 \\ 0 \end{pmatrix} =$

Solution: Note that $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector—in fact, it is $-\mathbf{v}_2$. Thus $\mathbf{A}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$.

Then note that $\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus $\mathbf{A}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \mathbf{A}\mathbf{v}_1 + \mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + 5\mathbf{v}_2 = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$.

b. [5 points] Sketch a qualitatively accurate phase portrait for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution: The equilibrium solution is $(0, 0)$, and there are two straight-line solutions, $\mathbf{v}_1$ and $\mathbf{v}_2$, both with outward trajectories. The eigenvalue associated with $\mathbf{v}_2$ is much larger than the other, so trajectories that are not on $k\mathbf{v}_1$ will bend to be parallel to $\mathbf{v}_2$, as shown in the figure below.



c. [5 points] Give two initial conditions for which the solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ will, as trajectories in the phase plane, eventually be parallel to the line $y = -x$. Give a short explanation of how you know your answer is correct.

Solution: Note that the line $y = -x$ is the line given by the second eigenvector. Any initial condition that does not lie on the first eigenvector will end up parallel to this, because the exponential for the second will dominate as time gets large. Thus two initial conditions that would work are $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (in the latter case the trajectory lies on the line).