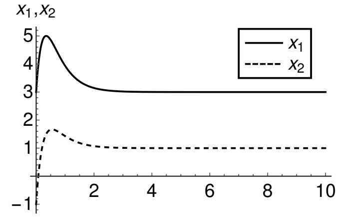
3. [14 points] Consider the systems of equations below. In each the vector $\mathbf{x}$ has components x_1 and x_2.

A. $\mathbf{x}' = \begin{pmatrix} 2 & -8 \\ 3 & -8 \end{pmatrix} \mathbf{x}$

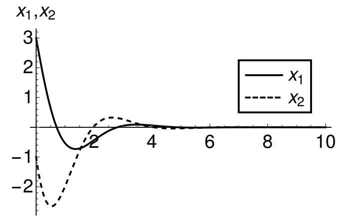
B. $\mathbf{x}' = \begin{pmatrix} 2 & -8 \\ 3 & -8 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Match each of these to one of the graphs to the right (note that two of these are component plots and two are phase portraits). Briefly explain how you know that your matching is correct.

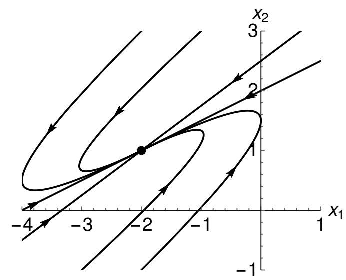
I.



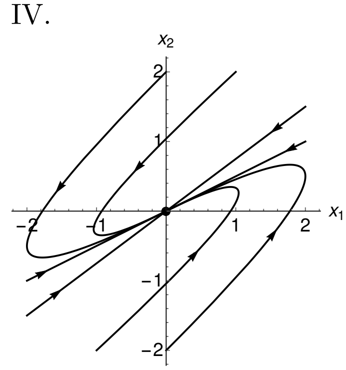
II.



III.



IV.



Solution: It is convenient to start by finding the eigenvalues of the coefficient matrix.

$$\det\left(\begin{pmatrix} 2-\lambda & -8 \\ 3 & -8-\lambda \end{pmatrix}\right) = \lambda^2 + 6\lambda + 8$$
$$= (\lambda+2)(\lambda+4) = 0,$$

so that $\lambda = -2$ and $\lambda = -4$. Thus in the phase plane, solutions for A must have two straight line trajectories through the origin (which could be graph IV, but not III); and component plots must go to $x_1 = 0 = x_2$ without oscillation (which excludes plots I and II). Thus A must match graph IV.

Next, B is the same system as A, but with forcing. The equilibrium solution for B is given by

$$2x_1 - 8x_2 = -2, 3x_1 - 8x_2 = 1,$$

so that, subtracting the first from the second, we have $x_1 = 3$. Plugging back into either equation, $x_2 = 1$. Therefore, one solution to B is $x_1 = 3$, $x_2 = 1$; this excludes graph III. For the same reason as for A, the component plots should not be oscillatory, so B must match graph I.