4. [15 points] For each of the following, find explicit, real-valued solutions, as indicated.

a. [7 points] Find the general solution to $y' = 7 - \dfrac{\cos(t)}{2+\sin(t)}y$.

Solution: This is linear, but not separable. Rewriting in standard form, we have $y' + \frac{\cos(t)}{2+\sin(t)}\,y = -7$, so that an integrating factor is $\mu = \exp(\int \frac{\cos(t)}{2+\sin(t)}\,dt) = \exp(\ln|2+\sin(t)|) = 2+\sin(t)$. Then $((2+\sin(t))\,y)' = 7(2+\sin(t))$, so that $(2+\sin(t))y = 7(2t-\cos(t))+C$, and

$$y = \frac{14t - 7\cos(t)}{2+\sin(t)} + \frac{C}{2+\sin(t)}.$$

b. [8 points] Solve the initial value problem: $\dfrac{y}{t^2+5}\,y' = 1$, $y(0) = -2$

Solution: This is nonlinear, and fortunately separable. Separating, we have $yy' = t^2+5$, so that $\frac{1}{2}\,y^2 = \frac{1}{3}\,t^3 + 5t + \tilde{C}$. Solving for y (and letting $C = 2\tilde{C}$),

$$y = \pm\sqrt{\frac{2}{3}\,t^3 + 10\,t + C}.$$

Applying the initial condition, we must take the negative square root and $C = 4$, so that

$$y = -\sqrt{\frac{2}{3}\,t^3 + 10\,t + 4}.$$