5. [14 points] The volumetric rate at which liquid leaves a cylindrical tank through a circular hole in its bottom is proportional to the square root of the volume of liquid in the tank. Suppose we have a cylindrical tank 5 meters tall with a 1 meter radius (so that its volume is 5π m^3) that is initially full of some liquid. At time $t = 0$, a circular hole opens in the base and more liquid is added at a rate of 1 m^2/hr.

a. [5 points] Write an initial value problem for the volume V of liquid in the tank. (Your answer will involve a constant of proportionality k.) Can you solve your equation? Explain. *(Do not actually solve the equation.)*

Solution: We have $\frac{dV}{dt}$ = (rate in) − (rate out) $= 1 - k\sqrt{V}$, with $V(0) = 5\pi$. This is nonlinear and separable (and thus solvable), but the resulting integral is not one that we are able to evaluate in closed form.

b. [5 points] Suppose that the solution to your equation in (a) is some function $V(t)$. If the liquid in the tank initially contains a particulate at a concentration of 1 g/m^3 and the liquid entering has a particulate concentration of 2 g/m^3, write an initial value problem for the amount of particulate in the tank. (Your answer will involve the unknown function $V(t)$.) Can you solve this equation? Explain. *(Do not actually solve the equation.)*

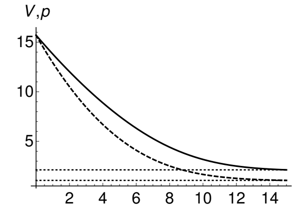
Solution: We can write an equation by considering the tank as a compartment: then $\frac{dp}{dt}$ = (rate in) − (rate out). The rate in is (1 m^3/hr)(2 g/m^3) = 2 (g/hr). The rate out is $(k\sqrt{V})(p(t)/V) = kp/\sqrt{V}$. Thus our differential equation is

$$\frac{dp}{dt} = 2 - \frac{k}{\sqrt{V}}\,p,$$

with initial condition $p(0) = 5\pi$. Assuming that we know the function $V(t)$, this is a linear (but not separable) equation, which we could solve with an integrating factor.

Problem 5, continued.

c. [4 points] What do you expect the long-term value for the volume $V(t)$ to be? Can you predict the long-term value for $p(t)$? If $k = 1$, which of the graphed functions to the right is $V(t)$ and which is $p(t)$? Why?



Solution: Note that the volume starts larger than 1, so initially $V' < 0$. There is an equilibrium volume, $k\sqrt{V} = 1$, so that $V = 1/k^2$, to which the volume will exponentially approach. In the long-term, then, we expect $V \to 1/k^2$, and therefore we expect the long-term dynamics of p to be given by $p' = f(p) = 2 - kp/\sqrt{1/k^2} = 2 - k^3 p$. The equilibrium value is $p_0 = 2/k^3$, and because $f'(p_0) = -k^3 < 0$, it is stable. We therefore expect that $p \to 2/k^3$. Finally, if $k = 1$, then $V \to 1$ and $p \to 2$, so the solid curve in the figure must be $p(t)$ and the dashed one $V(t)$.

6. [10 points] Consider the initial value problem $(1 - y^3)\frac{dy}{dt} = 1$, $y(0) = 0$.

a. [5 points] Without solving it, will this initial value problem have a unique solution?

Solution: Note that this is the same as $y' = f(y) = 1/(1 - y^3)$. The function f and its derivative f_y are discontinuous only if $y = 1$, so there will be a unique solution provided $y(0) \neq -1$. Thus we are guaranteed that there will be a unique solution.

b. [5 points] Solve the problem. Based on your solution, for what range of t and y values would you expect the solution to exist? Why?

Solution: Integrating both sides with respect to t, we have $y - \frac{1}{4}y^4 + C = t$, so that with $y(0) = 0$ we require $C = 0$ and have $t = y - \frac{1}{4}y^4$. Thinking of t as a function of y, this is a curve like a parabola opening down around the t-axis. Its vertex is when $\frac{d}{dy}(y - \frac{1}{4}y^4) = 1 - y^3 = 0$, or $y = 1$. When $y = 1$, $t = \frac{3}{4}$, and at that point the solution vanishes. Thus we have a (unique) solution only for $0 \le t < \frac{3}{4}$, $0 \le y < 1$. We may include $t = \frac{3}{4}$ and $y = 1$ in these, as the function continues to be well defined there.