7. [16 points] In lab we considered the van der Pol system $x' = y$, $y' = -x - \mu y \frac{df}{dx}$. Here, we suppose that $f'(x) = |x| - a$, so that this becomes $x' = y$, $y' = -x - \mu y\,(|x| - a)$.

a. [3 points] Find the critical point for this system.

Solution: Critical points are where $x' = y' = 0$, so from the first equation, $y = 0$. Then if $y = 0$ the second equation is $0 = -x$, so $x = 0$ also. The only critical point is $(0, 0)$.

b. [3 points] Linearize the system at your critical point from (a).

Solution: The only nonlinear term in the system is $y|x|$, which for x and y near $(0, 0)$ will be very small; thus, our linearization is

$$x' = y, \quad y' = -x + a\mu y.$$

c. [5 points] Suppose that your linear system from (b) is, for some k that depends on both of μ and a, $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix} \mathbf{x}$. Determine the type and stability of the critical point.

Solution: The coefficient matrix of the nonlinear system is $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix}$, so eigenvalues are given by $\det(\begin{pmatrix} -\lambda & 1 \\ -1 & k - \lambda \end{pmatrix}) = \lambda^2 - k\lambda + 1 = 0$. Thus, $\lambda = \frac{k}{2} \pm \frac{1}{2}\sqrt{k^2 - 4}$. This will have a positive real part if $k > 0$, so for $k > 0$ the origin is unstable while for $k < 0$ it is asymptotically stable. For $|k| < 2$, λ will be complex valued and we will have a spiral point; for $|k| > 2$, it will be a node. If $k = \pm 2$, we have a repeated eigenvalue and therefore the degenerate case with a single eigenvector.