1. [12 points] Suppose that a bucket with a capacity of 20 liters containing 1.3 kg of sand (which has a volume of 1 liter) is left outside in a very heavy rainstorm with a rainfall rate of 10 cm/hour. For a standard bucket, this results in water being added to the bucket at a rate of about 7 liters/hour.[1]

a. [6 points] Until the bucket fills, the amount of sand in the bucket is constant. Suppose that the rain fills the bucket before the end of the storm. Write an initial value problem for the amount of sand in the bucket after the bucket fills. You may take $t = 0$ as the time at which the bucket fills, and should assume that the sand is uniformly distributed through the water in the bucket.

Solution: Let x be the amount of sand in the bucket, in kg. Then $x(0) = 1.3$. No sand is added to the bucket, but it is lost as the bucket overflows. Water will leave the bucket at the same rate as the rain is entering the bucket, 7 liters/hr. There are 20 liters in the bucket, so the concentration of sand in the bucket is $\frac{x}{20}$ kg/liter, and the differential equation for x is

$$\frac{dx}{dt} = -7 \cdot \frac{x}{20} = -\frac{7}{20}\, x.$$

b. [6 points] What equilibrium solution, or solutions, does your equation in (a) have? Are they stable? Explain why this makes sense physically.

Solution: Equilibrium solutions are given by $\frac{dx}{dt} = f(x) = -\frac{7}{20}\, x = 0$, which is true if $x = 0$. Note that $f'(x) = -\frac{7}{20} < 0$, so this is a stable equilibrium. This makes sense: if it keeps raining forever, the sand will slowly get washed out of the bucket, leaving nothing but water.

[1] For those who prefer English units, this is, approximately, a 5 gallon bucket with a bit less than 3 lb, or a quarter gallon, of sand. The rainfall is about 4 in/hour.