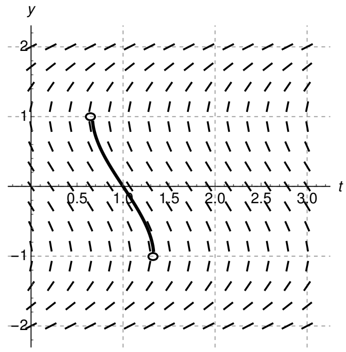
2. [15 points] Consider the direction field shown to the right, which corresponds to a first order differential equation $y' = f(t, y)$.



a. [5 points] Which of the following functions $f(t, y)$ is most likely to be the function in this differential equation? Briefly explain how you made your choice.

$f(t, y) = (y + 1)(y - 1)$ $\qquad$ $f(t, y) = \sin(\frac{\pi}{2}\, y)$

$\boxed{f(t, y) = \frac{2}{(y+1)(y-1)}}$ $\qquad$ $f(t, y) = \frac{2}{\sin(\frac{\pi}{2}\, y)}$

$f(t, y) = \frac{\sin(\frac{\pi}{2}\, t)}{y^2-1}$ $\qquad$ $f(t, y) = \frac{y+1}{y-1}$

Solution: We note that the direction field shows vertical slopes at $y = \pm 1$, which suggests that $f(t, y)$ is undefined there; this indicates that the answer is one of $f(t, y) = \frac{2}{(y+1)(y-1)}$, or $f(t, y) = \frac{\sin(\frac{\pi}{2}\, t)}{y^2-1}$. Further, the direction field is unchanged under translations in time, so it cannot be the latter. Therefore, the answer must be $f(t, y) = \frac{2}{(y+1)(y-1)}$.

b. [5 points] Sketch, on the direction field or below, the solution to $y' = f(t, y)$, $y(1) = 0$. For what values of t and y will it exist (you should be able to determine these without calculations)? Why?

Solution: We can trace a solution by using the fact that at every point in the (t, y) plane the solution must be parallel to the direction field ticks. This gives the result shown above. We note that at $y = \pm 1$ the solution curve reaches the point where its slope becomes undefined, and after that to follow the direction field we would end up with a curve that is not a function. Thus we expect that the solution will exist only for $-1 < y < 1$, which, from our sketch, is approximately $0.67 < t < 1.33$.

c. [5 points] Based on your choice of $f(t, y)$ in (a) and the corresponding direction field, are there any initial conditions (t_0, y_0) for which you cannot guarantee that there exists a unique solution? Explain.

Solution: We know that there will be a unique solution through any initial condition (t_0, y_0) where the function f and its partial derivative with respect to y, f_y, are continuous. From (a), this is whenever $y \neq \pm 1$, which corresponds in the direction field to where the slopes are defined (not vertical). Thus, for any initial condition (t_0, y_0) with $y_0 \neq \pm 1$ we will have a unique solution, but it may exist on a limited interval of time depending on whether or when it reaches one of the horizontal lines $y = \pm 1$. If we start with $y(t_0) = \pm 1$, the differential equation is not well defined, so it is not clear what we should do with it. If we assume that we can deal with that (e.g., by clearing the denominator), if $y_0 = \pm 1$ we might expect a non-unique solution.