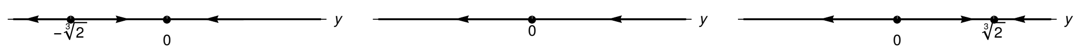
3. [15 points] Consider the equation $y' = ay - y^4$.

a. [5 points] Find all critical points for this equation.

Solution: Critical points are where $y' = 0 = y(a - y^3)$, so that $y = 0$ or $y = a^{1/3}$.

b. [6 points] Draw a phase line for each of the cases $a = -2, 0, 2$. Determine the stability of the critical points in each case.

Solution: If $a < 0$, $y' = -|a|y - y^4$, so that $y' < 0$ if $y < -|a^{1/3}|$, $y' > 0$ if $-|a^{1/3}| < y < 0$, and $y' < 0$ if $y > 0$. If $a = 0$, there is only one critical point, $y = 0$, and $y' = -y^4 < 0$ for all $y \neq 0$. If $a > 0$, $y' = ay - y^4 < 0$ if $y < 0$, $y' > 0$ if $0 < y < a^{1/3}$, and $y' < 0$ if $y > a^{1/3}$. This gives the three phase lines shown below.



$a = -2$ $a = 0$ $a = 2$

Thus, if $a < 0$, the critical point $y = a^{1/3}$ is unstable and $y = 0$ is asymptotically stable. If $a = 0$, $y = 0$ is unstable (or semi-stable). If $a > 0$, $y = a^{1/3}$ is asymptotically stable and $y = 0$ is unstable.

c. [4 points] Sketch a *bifurcation diagram* that shows the position of the critical points as a function of a in the ay-plane.

Solution: The bifurcation diagram is shown below. We indicate stable critical points with thick black curves (these are $y = 0$ for $a < 0$, and $y = a^{1/3}$ for $a > 0$, and unstable ones with dashed curves (these are $y = -|a|^{1/3}$ for $a < 0$ and $y = 0$ for $a > 0$).