4. [12 points] For each of the following give an example, as indicated. It may be useful to note that the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$ are $\lambda = 1$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda = 2$, $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a. [3 points] Give an example of a linear first-order equation that is not separable.

Solution: There are many possible answers; one is

$$y' + ty = \sin(t).$$

Note that any autonomous equation is necessarily separable.

b. [3 points] Give two distinct, non-zero solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ to the system $\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{x}$ for which $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is not a general solution to the system.

Solution: This just requires that we have two linearly dependent solutions. Using the eigenvalues above, one solution is $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ and a second, linearly dependent, solution is $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{2t}$.

c. [3 points] Give a 2×2 matrix $\mathbf{B}$, with all non-zero entries, for which $\mathbf{Bx} = \mathbf{0}$ has an infinite number of solutions.

Solution: Note that $\mathbf{x} = \mathbf{0}$ is always a solution, so we need only for $\mathbf{B}$ to be singular—that is, $\det(\mathbf{B}) = 0$. If $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this requires that $ac - bd = 0$. One such matrix is $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$.

d. [3 points] Give three different vectors, $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$, for which $\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{w}_j = k\,\mathbf{w}_j$, for some k. (The value of k need not be the same for all three vectors.)

Solution: This is just asking for an eigenvector-eigenvalue pair for the matrix. From above, we may use $k = 1$ and $\mathbf{w}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $k = 2$ and $\mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $k = 1$ or $k = 2$ and some multiple of the corresponding eigenvector, for example, $k = 1$ and $\mathbf{w}_3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. We could also take $k = 1$ and three different multiples of $\mathbf{v} = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$.