4. [12 points] For each of the following give an example, as indicated. It may be useful to note that the eigenvalues and eigenvectors of the matrix

\[
A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}
\]

are \( \lambda = 1, \ v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \) and \( \lambda = 2, \ v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

a. [3 points] Give an example of a linear first-order equation that is not separable.

Solution: There are many possible answers; one is

\[
y' + ty = \sin(t).
\]

Note that any autonomous equation is necessarily separable.

b. [3 points] Give two distinct, non-zero solutions \( x_1 \) and \( x_2 \) to the system

\[
x' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} x
\]

for which \( x = c_1 x_1 + c_2 x_2 \) is not a general solution to the system.

Solution: This just requires that we have two linearly dependent solutions. Using the eigenvalues above, one solution is \( x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \) and a second, linearly dependent, solution is \( x_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} \).

c. [3 points] Give a \( 2 \times 2 \) matrix \( B \), with all non-zero entries, for which \( Bx = 0 \) has an infinite number of solutions.

Solution: Note that \( x = 0 \) is always a solution, so we need only for \( B \) to be singular—that is, \( \det(B) = 0 \). If \( B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), this requires that \( ac - bd = 0 \). One such matrix is

\[
B = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.
\]

d. [3 points] Give three different vectors, \( w_1, w_2, \) and \( w_3 \), for which

\[
\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} w_j = k w_j,
\]

for some \( k \). (The value of \( k \) need not be the same for all three vectors.)

Solution: This is just asking for an eigenvector-eigenvalue pair for the matrix. From above, we may use \( k = 1 \) and \( w_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \), \( k = 2 \) and \( w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), and \( k = 1 \) or \( k = 2 \) and some multiple of the corresponding eigenvector, for example, \( k = 1 \) and \( w_3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \). We could also take \( k = 1 \) and three different multiples of \( v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \).