5. [16 points] Recall that the van der Pol equation we studied in lab 1 is given by

$$x'' + \mu f'(x)x' + x = 0,$$

or, as a system in x and $y = x'$,

$$x' = y, \qquad y' = -x - \mu f'(x)y,$$

for some function $f'(x)$. We assume that $\mu > 0$.

a. [3 points] Show that for any choice of $f'(x)$, the only critical point of the system formulation of the van der Pol equation is $(0, 0)$.

Solution: At a critical point, $x' = y' = 0$. Thus, from the first equation, $y = 0$. Then, from the second, $y' = 0 = -x - \mu f'(x) \cdot 0$, and $x = 0$ as well.

b. [4 points] Suppose that the Taylor expansion of $f'(x)$ around $x = 0$ is $f'(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots$. Use this expansion to linearize your system. Your linear system will involve the coefficients a_n.

Solution: Note that the first equation is already linear. The second is $y' = -x - \mu(a_0 + a_1 x + \cdots)y$, so that, dropping nonlinear terms, the linear system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -\mu a_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Problem 5, continued. We are considering the van der Pol system.

c. [5 points] Suppose that the system you obtained in (b) is $x' = -a_0\mu x - y$, $y' = x$. For what value or values of a_0 will the phase portrait for this system have only one straight-line of solution trajectories? No straight-line trajectories? Explain.

Solution: There is a single straight-line solution if the coefficient matrix has a repeated eigenvalue, and no straight-line solutions if the eigenvalues are complex. Here, eigenvalues are given by

$$\det(\begin{pmatrix} -\mu a_0 - \lambda & -1 \\ 1 & -\lambda \end{pmatrix}) = \lambda^2 + (\mu a_0)\lambda + 1 = (\lambda + \frac{1}{2}\mu a_0)^2 + 1 - \frac{1}{4}\mu^2 a_0^2 = 0.$$

This gives $\lambda = -\frac{1}{2}\mu a_0 \pm \frac{1}{2}\sqrt{\frac{1}{4}\mu^2 a_0^2 - 1}$. Thus, there will be a single straight-line solution if $a_0 = \pm\frac{2}{\mu}$, and no straight-line solutions if $|a_0| < \frac{2}{\mu}$, so that $\frac{1}{4}\mu a_0^2 - 1 < 0$.

d. [4 points] When a_0 is picked so that there is a single straight-line of solution trajectories in the phase portrait for this system, give an initial condition that will result in a straight-line trajectory in the phase plane.

Solution: This is when $a_0 = \pm\frac{2}{\mu}$, so that $\lambda = -\frac{1}{2}\mu a_0 = \mp 1$. If $a_0 = \frac{2}{\mu}$, the eigenvector satisfies the equation $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus any initial condition on the line $y = -x$ will converge along that line to the origin.