6. [15 points] Find real-valued solutions to each of the following, as indicated. Where possible, find explicit solutions.

a. [7 points] Find the general solution to the Gompertz equation, $\frac{dy}{dt} = r\,y\,\ln(\frac{K}{y})$.

Solution: We rewrite this using rules of logarithms as $\frac{dy}{dt} = ry(\ln(K) - \ln(y))$, and separate to have

$$\frac{dy/dt}{y(\ln(y) - \ln(K))} = -r.$$

Integrating both sides, we have

$$\ln|\ln(y) - \ln(K)| = -rt + \hat{C},$$

so that $\ln(y) - \ln(K) = Ce^{-rt}$ (with $C = \pm e^{\hat{C}}$). Adding $\ln(K)$ to both sides and exponentiating again,

$$y = Ke^{C\exp(-rt)}.$$

b. [8 points] Solve $y' = 3t - \frac{t}{1+t^2}\,y$, with $y(0) = 3$.

Solution: This is linear and not separable. We rewrite it in standard form to get $y' + \frac{t}{1+t^2}y = 3t$, and an integrating factor is $\mu = \exp(\int t/(1+t^2)\,dt) = \exp(\ln(1+t^2)/2 = \sqrt{1+t^2}$. Multiplying by μ, we have

$$\left(y\,\sqrt{1+t^2}\right)' = 3t\sqrt{1+t^2},$$

so that on integrating both sides, $y\,\sqrt{1+t^2} = (1+t^2)^{3/2} + C$, and

$$y = 1 + t^2 + \frac{C}{\sqrt{1+t^2}}.$$

With $y(0) = 3$, we have $y(0) = 3 = 1 + C$, and $C = 2$, so

$$y = 1 + t^2 + \frac{2}{\sqrt{1+t^2}}.$$