7. [15 points] Find explicit, real-valued solutions to each of the following, as indicated.

a. [7 points] Find the general solution to the system $x' = y$, $y' = 2x + y$.

Solution: Written as a matrix system, the coefficient matrix on the right-hand side is $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$, so that eigenvalues satisfy

$$\det(\begin{pmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{pmatrix}) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0.$$

Thus $\lambda = -1$ or $\lambda = 2$. If $\lambda = -1$, the eigenvector satisfies $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ (or any nonzero constant multiple of this). If $\lambda = 2$, we have $\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$ (or any constant multiple thereof). Thus the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} \\ -c_1 e^{-t} + 2c_2 e^{2t} \end{pmatrix}.$$

b. [8 points] Solve $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Solution: Eigenvalues of the coefficient matrix are given by

$$\det(\begin{pmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{pmatrix}) = \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1 = 0.$$

Thus $\lambda = 1 \pm i$. We consider $\lambda = 1 + i$, and use it to obtain two linearly independent real-valued solutions. In this case, the eigenvector satisfies $\begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so that one choice is $\mathbf{v} = \begin{pmatrix} 1 & 1+i \end{pmatrix}^T$. (Using the second equation gives $\mathbf{v} = \begin{pmatrix} 1-i & 2 \end{pmatrix}^T$.) We write the corresponding complex-valued solution and obtain real-valued solutions by taking the real and imaginary parts:

$$\mathbf{x} = \begin{pmatrix} e^t(\cos(t) + i\sin(t)) \\ e^t(\cos(t) + i\sin(t) + i\cos(t) - \sin(t)) \end{pmatrix},$$

so that a general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} \cos(t) \\ \cos(t) - \sin(t) \end{pmatrix} e^t + c_2 \begin{pmatrix} \sin(t) \\ \cos(t) + \sin(t) \end{pmatrix} e^t.$$

We note that if $c_1 = 0$, $c_2 = 2$ the initial condition is satisfied; thus $\mathbf{x} = 2 \begin{pmatrix} \sin(t) \\ \cos(t) + \sin(t) \end{pmatrix} e^t$. With the second $\mathbf{v}$, above, the complex-valued form of and general solution for $\mathbf{x}$ are $\mathbf{x} = \begin{pmatrix} e^t(\cos(t) + i\sin(t) - i\cos(t) + \sin(t)) \\ e^t(2\cos(t) + 2i\sin(t)) \end{pmatrix}$ and $\mathbf{x} = c_1 \begin{pmatrix} \cos(t) + \sin(t) \\ 2\cos(t) \end{pmatrix} e^t + c_2 \begin{pmatrix} -\cos(t) + \sin(t) \\ 2\sin(t) \end{pmatrix} e^t$, respectively. In this case $c_1 = c_2 = 1$ to give the solution above.