6. [14 points] Consider a chemical reaction in which two chemicals \( X \) and \( Y \) combine to form a new compound \( Z \). We write \( X + Y \rightarrow Z \). Then the speed of the reaction (that is, the rate at which the compound \( Z \) appears) is proportional to product of the concentrations of the compounds \( X \) and \( Y \). Because one molecule of each of \( X \) and \( Y \) are used for each molecule of \( Z \) that is created, this results in the differential equation

\[
\frac{dz}{dt} = \alpha (x_0 - z)(y_0 - z),
\]

where \( z \) is the concentration of \( Z \), \( \alpha \) is the rate constant for the reaction and \( x_0 \) and \( y_0 \) are the initial concentrations of \( X \) and \( Y \). If we initially have none of compound \( Z \), the initial condition is \( z(0) = 0 \).

(a) [7 points] Suppose that \( 0 < \alpha < 1 \) and \( 0 < x_0 < y_0 \). Without solving the equation, determine what you expect the long-term concentration of \( Z \) will be by doing a qualitative analysis of the given equation. (While you may confirm your conclusions by speaking to the chemistry, your answer should be grounded in the analysis of the differential equation.)

(b) [7 points] Now suppose that \( 0 < \alpha < 1 \) and \( x_0 = y_0 > 0 \). How does your analysis of the equation from (a) change? Explain by doing a similar analysis.