2. [15 points] Solve each of the following, finding explicit real-valued solutions as indicated.

a. [8 points] Find the solution to the initial value problem $x' = x + 2y$, $y' = 4x + 3y$, $x(0) = -1$, $y(0) = 8$.[1]

Solution: In matrix form, this is $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. The eigenvalues of the coefficient matrix are given by $\det(\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}) = \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1) = 0$. Thus $\lambda = 5$ or $\lambda = -1$. If $\lambda = 5$, eigenvectors satisfy $\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Similarly, if $\lambda = -1$, eigenvectors satisfy $\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}.$$

Applying the initial conditions, we have $c_1 - c_2 = -1$ and $2c_1 + c_2 = 8$. Adding the two equations, $3c_1 = 7$, so $c_1 = \frac{7}{3}$ and $c_2 = \frac{10}{3}$. Thus

$$\mathbf{x} = \frac{7}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + \frac{10}{3} \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-t}.$$

b. [7 points] Find the general solution to $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Solution: The eigenvalues of the coefficient matrix are given by $(2-\lambda)(-\lambda) + 2 = \lambda^2 - 2\lambda + 2 = (\lambda-1)^2 + 1 = 0$. Thus $\lambda = 1 \pm i$. If $\lambda = 1 + i$, the components of the eigenvector satisfy (from the second row of the coefficient matrix) $v_1 - (1+i)v_2 = 0$, so we may take $\mathbf{v} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$. A complex-valued solution is therefore $\mathbf{x} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} e^t(\cos(t) + i\sin(t))$. Separating the real and imaginary parts of this, we have

$$\mathbf{x} = c_1 \begin{pmatrix} \cos(t) - \sin(t) \\ \cos(2t) \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos(t) + \sin(t) \\ \sin(t) \end{pmatrix} e^t.$$

Alternately, we could take $\mathbf{v} = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$, so that

$$\mathbf{x} = c_1 \begin{pmatrix} 2\cos(t) \\ \cos(t) + \sin(t) \end{pmatrix} e^t + c_2 \begin{pmatrix} 2\sin(t) \\ -\cos(t) + \sin(t) \end{pmatrix} e^t.$$

[1] The original exam copy had $y'(0) = 8$; a correct solution may be obtained applying this as well.