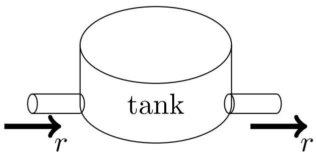
3. [14 points] Consider a storage tank containing V_0 liters of pure water, having an open top, as suggested in the figure to the right. Water containing a chemical at a concentration of c_0 kg/liter enters the tank at a rate r liters/min. The well-mixed solution leaves the tank at the same rate.



a. [5 points] Write down an initial value problem for the amount A of the chemical P in the tank.

Solution: We have $A' = (\text{rate in}) - (\text{rate out})$. The rate in is c_0 kg/liter $\cdot\, r$ liter/min; the rate out is (A/V_0) kg/liter $\cdot\, r$ liter/min. Thus $A' = c_0 r - \frac{r}{V_0} A$. The initial condition is that the tank initially does not contain any of the chemical, so $A(0) = 0$.

b. [5 points] Now suppose that the liquid can evaporate from the top of the tank. This results in a loss proportional to the surface area, so the volume of liquid in the tank decreases by $\alpha\pi a^2$ liters/min, where a is the radius of the (cylindrical) tank. Write (but do not solve) a new differential equation for the amount of chemical in the tank. You should assume that the chemical does not evaporate as well. Be sure that it is clear why your equation has the form it does.

Solution: Note that the volume in the tank is now not constant. Because the evaporation rate does not depend on the volume, and the tank is assumed cylindrical, the loss rate does not change with time, and we have $V = V_0 - \alpha\pi a^2\, t$. Thus our new equation is

$$A' = c_0 r - \frac{r}{V_0 - \alpha\pi a^2\, t} A.$$

c. [4 points] Consider your differential equation in (b) with the initial condition $A(0) = c_0 V_0$. On what range of t values, if any, is a unique solution for A guaranteed to exist? Explain.

Solution: We note that the equation we have is linear, so we are guaranteed by our existence and uniqueness theorems that we will have a unique solution whereever the coefficients are continuous. This will be the case for $0 \le t < \frac{V_0}{\alpha\pi a^2}$.

