4. [15 points] In lab 2 we considered the van der Pol equation, $x'' + \mu(x^2 - 1)x' + x = 0$. We consider this equation in this problem.

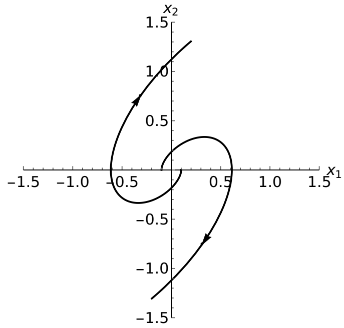
a. [4 points] Write the van der Pol equation as a system of two first-order differential equations and show that the only critical point of the system is $(0, 0)$.

Solution: Letting $x_1 = x$ and $x_2 = x'$, we have

$$x_1' = x_2,$$
$$x_2' = -x_1 - \mu(x_1^2 - 1)x_2.$$

Critical points occur when $x_1' = x_2' = 0$. If $x_1' = 0$, we know that $x_2 = 0$. Then the second equation gives $x_2' = -x_1 = 0$, so the only critical point is $(0, 0)$.

b. [6 points] Let $\mu = 1$. As we saw in lab, the linearization of the system you obtained in (a) at the critical point is then $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}$. Solve this system and sketch a phase portrait for it.



Solution: Let $\mathbf{x} = \mathbf{v}e^{\lambda t}$. The λ are the eigenvalues of the coefficient matrix, which satisfy $\lambda^2 - \mu\lambda + 1 = 0$, so $\lambda = \frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4}$. Because $0 < \mu < 2$, this is $\lambda = \frac{\mu}{2} \pm \frac{1}{2}i\sqrt{4 - \mu^2}$, and trajectories in the phase plane are outward spirals. Noting that at $(1, 0)$, $x_1' = 0$ and $x_2' = -1$, we conclude that they are clockwise spirals, as shown in the figure to the right.

Completing the solution of the system, note that the first row of the equation for $\mathbf{v}$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, gives $\mathbf{v} = \begin{pmatrix} 1 & \lambda \end{pmatrix}^T$. Separating the real and imaginary parts of the solution $\mathbf{x} = \mathbf{v}e^{\lambda t}$ allows us to write, with $\omega = \frac{1}{2}\sqrt{4 - \mu^2}$,

$$\mathbf{x} = c_1 \begin{pmatrix} \cos(\omega t) \\ \frac{\mu}{2}\cos(\omega t) - \omega\sin(\omega t) \end{pmatrix} e^{\mu t/2} + c_2 \begin{pmatrix} \sin(\omega t) \\ \omega\cos(\omega t) + \frac{\mu}{2}\sin(\omega t) \end{pmatrix} e^{\mu t/2}.$$

c. [5 points] Explain what your solution in (b) tells about solutions to the original van der Pol equation. Then, using your system from (a), find the slope of the trajectory in the phase plane at $(3, -\frac{3}{8})$. Explain what these tell you about how the phase portrait for (a) is different from that for (b), and how this is related to your work in lab.

Solution: Because this is a linearization at $(0, 0)$, we expect the nonlinear van der Pol equation to have this behavior when x is small. That is, if we start with a small initial condition for x we initially expect the solution to be an oscillatory function with increasing magnitude.

Plugging $(-3, -\frac{3}{8})$ into the nonlinear system from (a), we have $(x_1', x_2') = (-\frac{3}{8}, 0)$. Thus at this point trajectories are moving *inward*. This is different from the linear behavior, and indicates that farther away from $(0, 0)$ the nonlinear effects become significant, which results in trajectories from near $(0, 0)$ and far from $(0, 0)$ converging to the limit cycle that we saw in lab.