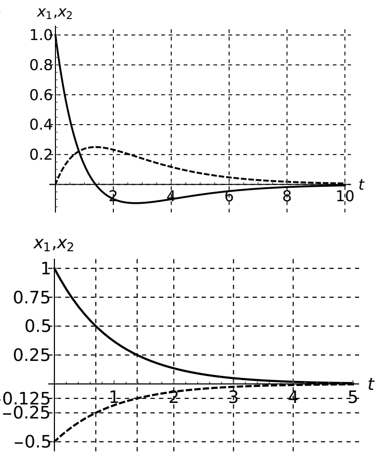
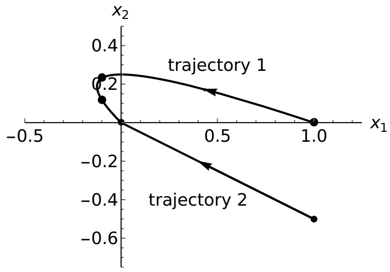
5. [15 points] The following considers the solution (x_1, x_2) to a linear system of two first-order constant coefficient equations, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.



a. [5 points] If the solutions to this system for two different initial conditions are shown to the right (in both graphs, the solid curve is x_1 and the dashed curve is x_2), sketch the corresponding trajectories in the phase plane. Label each trajectory.

Solution: For the first trajectory, we note that we start at $(1, 0)$ in the phase plane. At $t = 2$ we are approximately at $(-0.1, 0.2)$, and at $t = 4$, at $(-0.1, 0.1)$. Drawing a smooth curve through these, estimating more points from the graph shown, we get trajectory 1 in the lower graph (which shows the points at $t = 0$, $t = 2$, and $t = 4$ as well). For the second trajectory the provided t values are irregular, but give the points $(1, -0.5)$, $(0.5, -0.25)$, and $(0.25, -0.125)$. Thus, this appears to be a linear trajectory with slope $m = -0.5$, the lower trajectory shown in the figure to the right.



b. [5 points] Given your trajectories in (a), give possible values for the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. Be sure that it is clear how you obtain your answer.

Solution: The linear trajectory tells us one eigenvector: $\mathbf{v}_1 = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$, and we know that the associated eigenvalue must be negative. Because we have a second trajectory that collapses to the origin both eigenvalues must be negative, and all eigenvalues and eigenvectors must be real. The values of the eigenvalues, their relative magnitude, and the second eigenvector (if any) are not definitively specified given the accuracy that we are able to produce in our graph above, though if there is a second eigenvector it clearly must have negative slope to avoid intersecting trajectory 1.

With some thought we might conclude that it must have a slope close to or less (more negative) than the first to avoid an intersection with trajectory 1 as $t \to -\infty$, but this is a subtle point—at the expected accuracy of the figure generated in (a) this observation may be difficult to make.

If there is a second eigenvector with slope less than $m = -\frac{1}{2}$, we could take $\mathbf{v}_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$. In this case trajectory 1 in the graph above requires that the eigenvalue associated with $\mathbf{v}_1$ be less than that associated with $\mathbf{v}_2$, e.g., $\lambda_1 = -2$ and $\lambda_2 = -1$. (If we picked $\mathbf{v}_2$ to have slope slightly larger than $m = -\frac{1}{2}$ the order of the eigenvalues would have to reverse for the trajectory to behave as shown.)

c. [5 points] Sketch a phase portrait for the system given your answer to (b). *(If you were unable to complete (b), assume that your eigenvalues and eigenvectors are* $\lambda = -2$ *with* $\mathbf{v} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ *and* $\lambda = -1$ *with* $\mathbf{v} = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$*.)*

Solution: With the eigenvectors indicated above, we have two straight line solutions, $y = -x/2$ and $y = -x$. Trajectories collapse fastest along the first, giving the phase portrait to the left, below. If we used the eigenvalues and eigenvectors in the parenthetical note, trajectories collapse first to the other eigenvector and we would have the figure to the right.