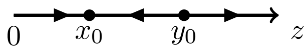
6. [14 points] Consider a chemical reaction in which two chemicals X and Y combine to form a new compound Z. We write $X + Y \to Z$. Then the speed of the reaction (that is, the rate at which the compound Z appears) is proportional to product of the concentrations of the compounds X and Y. Because one molecule of each of X and Y are used for each molecule of Z that is created, this results in the differential equation

$$\frac{dz}{dt} = \alpha(x_0 - z)(y_0 - z),$$

where z is the concentration of Z, α is the rate constant for the reaction and x_0 and y_0 are the initial concentrations of X and Y. If we initially have none of compound Z, the initial condition is $z(0) = 0$.

a. [7 points] Suppose that $0 < \alpha < 1$ and $0 < x_0 < y_0$. Without solving the equation, determine what you expect the long-term concentration of Z will be by doing a qualitative analysis of the given equation. (While you may confirm your conclusions by speaking to the chemistry, your answer should be grounded in the analysis of the differential equation.)

Solution: We see that equilibrium solutions are $z = x_0$ and $z = y_0$. The right-hand side of the equation is an upward opening parabola, so we will have the phase line shown below.



This indicates that the critical point $z = x_0$ is stable, and this is the long-term expected concentration of Z provided $z(0) < y_0$. As the reaction is purported to create Z, we expect $z(0) = 0$, so that $z \to x_0$. (At this point all of the chemical X is used up, so that $x = 0$, and we will have $y = y_0 - x_0$. We note that physically we are unable to create amounts of Z that are greater than either of x_0 or y_0.)

b. [7 points] Now suppose that $0 < \alpha < 1$ and $x_0 = y_0 > 0$. How does your analysis of the equation from (a) change? Explain by doing a similar analysis.

Solution: Now there is a single equilibrium solution, x_0, which is semi-stable (that is, unstable). However, because we do not expect $z > x_0$ at any time, we expect the same long-term behavior: $z \to x_0$. This is illustrated in the phase line, shown below.