6. [14 points] Consider a chemical reaction in which two chemicals $X$ and $Y$ combine to form a new compound $Z$. We write $X + Y \rightarrow Z$. Then the speed of the reaction (that is, the rate at which the compound $Z$ appears) is proportional to product of the concentrations of the compounds $X$ and $Y$. Because one molecule of each of $X$ and $Y$ are used for each molecule of $Z$ that is created, this results in the differential equation

$$\frac{dz}{dt} = \alpha (x_0 - z)(y_0 - z),$$

where $z$ is the concentration of $Z$, $\alpha$ is the rate constant for the reaction and $x_0$ and $y_0$ are the initial concentrations of $X$ and $Y$. If we initially have none of compound $Z$, the initial condition is $z(0) = 0$.

**a. [7 points]** Suppose that $0 < \alpha < 1$ and $0 < x_0 < y_0$. Without solving the equation, determine what you expect the long-term concentration of $Z$ will be by doing a qualitative analysis of the given equation. (While you may confirm your conclusions by speaking to the chemistry, your answer should be grounded in the analysis of the differential equation.)

**Solution:** We see that equilibrium solutions are $z = x_0$ and $z = y_0$. The right-hand side of the equation is an upward opening parabola, so we will have the phase line shown below.

This indicates that the critical point $z = x_0$ is stable, and this is the long-term expected concentration of $Z$ provided $z(0) < y_0$. As the reaction is purported to create $Z$, we expect $z(0) = 0$, so that $z \rightarrow x_0$. (At this point all of the chemical $X$ is used up, so that $x = 0$, and we will have $y = y_0 - x_0$. We note that physically we are unable to create amounts of $Z$ that are greater than either of $x_0$ or $y_0$.)

**b. [7 points]** Now suppose that $0 < \alpha < 1$ and $x_0 = y_0 > 0$. How does your analysis of the equation from (a) change? Explain by doing a similar analysis.

**Solution:** Now there is a single equilibrium solution, $x_0$, which is semi-stable (that is, unstable). However, because we do not expect $z > x_0$ at any time, we expect the same long-term behavior: $z \rightarrow x_0$. This is illustrated in the phase line, shown below.

0 $x_0$ $y_0$ $z$