7. [12 points] Each of the following has an answer that you can determine with minimal work. In each, $\mathbf{A}$ is a 2×2 real-valued matrix (but in each is a different matrix). Provide the answer, and give a two sentence explanation of how you obtained it.

a. [4 points] If $\mathbf{A}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$ and eigenvalues of $\mathbf{A}$ are λ_1 and λ_2, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix}3\\4\end{pmatrix}$ is

$\mathbf{x} = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} - \begin{pmatrix}1\\2\end{pmatrix}$

Solution: We see that $\mathbf{x}_c = -\begin{pmatrix}1\\2\end{pmatrix}$ is the equilibrium solution for the given equation. Letting $\mathbf{x} = \mathbf{u} + \mathbf{x}_c$ will result in $\mathbf{u}$ solving the homogeneous equation $\mathbf{u}' = \mathbf{A}\mathbf{u}$, so that $\mathbf{u} = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t}$, which gives the indicated general solution.

b. [4 points] If the only eigenvalue of $\mathbf{A}$ is $\lambda = -3$, with only one eigenvector, $\mathbf{v} = \begin{pmatrix}4\\3\end{pmatrix}$, then as $t \to \infty$, the largest term in all solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ will be

$\mathbf{x} = c\,t\,\mathbf{v}e^{-3t} = c\,t\begin{pmatrix}4\\3\end{pmatrix}e^{-3t}$

Solution: The general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is $\mathbf{x} = c_1(\mathbf{v}t + \mathbf{w})e^{\lambda t} + c_2\mathbf{v}e^{\lambda t}$. The dominant term in this expression is $\mathbf{x} = c_1\mathbf{v}te^{\lambda t}$.

c. [4 points] If the eigenvalues of $\mathbf{A}$ are $\lambda = -3$ and $\lambda = 5$, with eigenvectors $\mathbf{v} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}-1\\1\end{pmatrix}$, then the number of solutions $\mathbf{x}$ to $\mathbf{A}\mathbf{x} = \mathbf{0}$ is

Exactly one: $\mathbf{x} = \mathbf{0}$.

and the number of solutions to $\mathbf{A}\mathbf{x} = 3\mathbf{x}$ is

Exactly one.

Solution: Note that if $\mathbf{A}$ doesn't have an eigenvalue $\lambda = 0$, then its determinant must be nonzero, so $\mathbf{A}\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$. Similarly, $\mathbf{A}\mathbf{x} = k\mathbf{x}$ has multiple solutions only if k is an eigenvalue, which isn't the case here.