2. (a) (4 points) The temperature T_R of a certain nuclear reactor is decaying exponentially in time toward a background temperature $T_B = 25^\circ$ C: $T_R = T_B + ae^{-t}$ where t is measured in hours and a is a constant. The temperature T of a container of water placed in the reactor is subject to Newton's Law of Cooling/Heating: the time rate of change of the temperature T is proportional to the difference between T and the reactor temperature. If at time $t = 0$,

- The reactor temperature is at 325° C,
- The water temperature is the same as the background temperature, and
- The water temperature is increasing by 600° C per hour,

find (but do not solve) the precise first-order ODE satisfied by the water temperature $T(t)$.

Solution: To have $T_R = 325^\circ$ C at $t = 0$ requires that $a = 300^\circ$ C, so the time-dependent reactor temperature is $T_R = 25 + 300e^{-t}$. Newton's Law of Cooling/Heating says that there is some real constant k of proportionality so that

$$\frac{dT}{dt} = k(T - T_R(t)) = k(T - 25 - 300e^{-t}).$$

Putting $t = 0$, $T = 25^\circ$ C, and $dT/dt = 600^\circ$ C per hour then gives $600 = -300k$ so $k = -2$. Therefore the differential equation for $T(t)$ is

$$\boxed{\frac{dT}{dt} = -2(T - 25 - 300e^{-t}).}$$

(b) (4 points) The number $F(t)$ of fish in a small lake t weeks after the beginning of summer is governed by the differential equation $F' = -4F + S(t)$, where $S(t)$ (fish/week) is the rate at which the lake is stocked with fish by the park ranger. The park ranger starts out putting in 1000 fish/week but gets busy with other things and ends up adding fewer fish every week so that $S(t) = 1000e^{-2t}$. If there aren't any fish in the lake at the beginning of summer, find $F(t)$.

Solution: An integrating factor is e^{4t}, so the equation becomes

$$\frac{d}{dt}(e^{4t}F(t)) = 1000e^{2t} \implies e^{4t}F(t) = 500e^{2t} + C$$

where C is a constant of integration. Putting $F(0) = 0$ and setting $t = 0$ gives $C = -500$.

$$\boxed{F(t) = 500e^{-2t} - 500e^{-4t}.}$$

An alternate approach that is equally effective and straightforward is to use the variation of parameters method from Written HW 1.