4. (a) (4 points) Solve the initial-value problem $x' = x^2/t + 3x^2t^2$, $x(-1) = \frac{1}{2}$, for $x = x(t)$.

Solution: separating the variables gives $x^{-2}\,\mathrm{d}x = (t^{-1} + 3t^2)\,\mathrm{d}t$ so $-x^{-1} = \ln(|t|) + t^3 + C$. Putting in $(t, x) = (-1, \frac{1}{2})$ gives $C = -1$. So

$$\boxed{x(t) = \frac{1}{1 - t^3 - \ln(|t|)} = \frac{1}{1 - t^3 - \ln(-t)}.}$$

(b) (4 points) A general solution of the differential equation $x' = t/x$ for $x = x(t)$ has the implicit form $x^2 - t^2 = C$. Find the (maximal) interval of existence of the solution with initial condition $x(5) = 4$.

Solution: putting $t = t_0 = 5$ and $x = x_0 = 4$ gives $C = -9$. Taking square roots, there are then two possible solutions for this value of C, namely $x(t) = \pm\sqrt{t^2 - 9}$. To have $x(5) = 4$ then requires picking the positive sign so $x(t) = \sqrt{t^2 - 9}$. This is real and differentiable for $t < -3$ and for $t > 3$, only the latter of which is an interval containing $t_0 = 5$. So the interval of existence is

$$\boxed{(3, +\infty).}$$