4. (a) (4 points) Solve the initial-value problem \( x' = \frac{x^2}{t} + 3x^2t^2, \ x(-1) = \frac{1}{2}, \) for \( x = x(t). \)

Solution: separating the variables gives \( x^{-2} \, dx = (t^{-1} + 3t^2) \, dt \) so \( -x^{-1} = \ln(|t|) + t^3 + C. \)

Putting in \( (t, x) = (-1, \frac{1}{2}) \) gives \( C = -1. \) So

\[
x(t) = \frac{1}{1 - t^3 - \ln(|t|)} = \frac{1}{1 - t^3 - \ln(-t)}.
\]

(b) (4 points) A general solution of the differential equation \( x' = \frac{t}{x} \) for \( x = x(t) \) has the implicit form \( x^2 - t^2 = C. \) Find the (maximal) interval of existence of the solution with initial condition \( x(5) = 4. \)

Solution: putting \( t = t_0 = 5 \) and \( x = x_0 = 4 \) gives \( C = -9. \) Taking square roots, there are then two possible solutions for this value of \( C, \) namely \( x(t) = \pm \sqrt{t^2 - 9}. \) To have \( x(5) = 4 \) then requires picking the positive sign so \( x(t) = \sqrt{t^2 - 9}. \) This is real and differentiable for \( t < -3 \) and for \( t > 3, \) only the latter of which is an interval containing \( t_0 = 5. \) So the interval of existence is

\[ (3, +\infty). \]