6. Consider the system with parameter $h$

$$\mathbf{x}' = \begin{pmatrix} -3 & 1 \\ -1 & h \end{pmatrix} \mathbf{x}$$

for a vector function $\mathbf{x} = \mathbf{x}(t)$.

(a) (4 points) For which value(s) of $h$ is there a solution of this system of the form

$$\mathbf{x}(t) = \begin{pmatrix} (at+b)\mathrm{e}^{-4t} \\ (t+c)\mathrm{e}^{-4t} \end{pmatrix}$$

for some constants $a, b, c$? (No need to find $a, b, c$, just $h$.)

Solution: We need that the coefficient matrix has a repeated eigenvalue $\lambda = -4$. The characteristic equation for the problem reads $(h-\lambda)(-3-\lambda)+1 = 0$ or $\lambda^2 + (3-h)\lambda + 1 - 3h = 0$. The discriminant is $(3-h)^2 - 4 + 12h = h^2 + 6h + 5 = (h+5)(h+1)$. So we have a repeated root if either $h = -5$ or $h = -1$. In those cases, the repeated root is $\lambda = (h-3)/2$ which is $-4$ only for $h = -5$ (it is $-2$ if $h = -1$ instead). Therefore the only possible value of $h$ is

$$\boxed{h = -5.}$$

Another approach would be to first insist that $\lambda = -4$ is an eigenvalue. Since the characteristic equation is linear in $h$, this automatically and immediately yields $h = -5$. However, one should then confirm that this eigenvalue is repeated when $h = -5$ because otherwise the solution does not include any term proportional to $t\mathrm{e}^{-4t}$. A third approach is to deduce two conditions (for $\lambda = -4$ to be an eigenvalue and for it to be repeated) by simply equating the characteristic polynomial to $(\lambda+4)^2$, and then check that all the coefficients of powers of $\lambda$ match if and only if $h = -5$.

(b) (4 points) Suppose that $h = -3$. Solve the initial-value problem for the system with initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution: The characteristic equation reads $\lambda^2 + 6\lambda + 10 = 0$ which has complex-conjugate roots $\lambda = -3 \pm \mathrm{i}$. An eigenvector for $\lambda = -3 + \mathrm{i}$ is $\mathbf{v} = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$, so the general solution is $c_1\mathbf{u}(t) + c_2\mathbf{w}(t)$, where $\mathbf{u}(t)$ and $\mathbf{w}(t)$ are the real and imaginary parts of the vector solution $\mathrm{e}^{-3t}\mathrm{e}^{\mathrm{i}t}\mathbf{v} = \mathrm{e}^{-3t}(\cos(t) + \mathrm{i}\sin(t))\mathbf{v}$. Therefore $\mathbf{u}(t) = \mathrm{e}^{-3t}\begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$ and $\mathbf{w}(t) = \mathrm{e}^{-3t}\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$. At $t = 0$ the general solution reads $c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so we need $c_1 = c_2 = 1$. The solution is therefore

$$\boxed{\mathbf{x}(t) = \mathrm{e}^{-3t}\begin{pmatrix} \cos(t) + \sin(t) \\ \cos(t) - \sin(t) \end{pmatrix}.}$$