8. True or false? Write out the full word "true" or "false" and provide a brief justification (2 points each).

   (a) $\mu = 2$ is a bifurcation point for the equation $y' = y^5 + \mu y^4 + y^3$.

   True. Since $y^5 + \mu y^4 + y^3 = y^3(y^2 + \mu y + 1)$, the critical points are $y = 0$ and $y = \frac{1}{2}(-\mu \pm \sqrt{\mu^2 - 4})$ provided the latter two are real. So there is just one equilibrium if $-2 < \mu < 2$ and three equilibria if $\mu < -2$ or $\mu > 2$.

   (b) There is exactly one differentiable function $y(t)$ defined near $t = 1$ whose graph passes through the point $(1, 2)$ and such that $y(t)y'(t)$ and $1 + y(t)\sin(t)$ are actually the same functions of t.

   True. This is the same question as whether there exists a unique solution of the initial-value problem $y' = f(t, y)$ where $f(t, y) := y^{-1} + \sin(t)$ with initial condition $y(1) = 2$. Since $f_y(t, y) = -y^{-2}$, we see that $f(t, y)$ and $f_y(t, y)$ are continuous on the whole (t, y)-plane except where $y = 0$. The point $(1, 2)$ is not on the horizontal line $y = 0$, so we can center it within a rectangle like $0 < t < 2$ and $1 < y < 3$ on which both functions are continuous. So by Theorem 2.4.2 there is a unique solution.

   (c) There is a matrix $\mathbf{A}$ with the following eigenvalues/eigenvectors:

$$\lambda_1 = 2,\ \mathbf{x}_1 = \begin{pmatrix} -20 \\ 25 \end{pmatrix}; \quad \lambda_2 = 65,\ \mathbf{x}_2 = \begin{pmatrix} 40 \\ -50 \end{pmatrix}.$$

   False. This matrix has distinct eigenvalues, but the given eigenvectors are proportional, which is not possible according to Theorem 3.1.3.

   (d) There is some continuous function $f(y)$ for which $y' = f(y)$ has only two equilibria, both unstable.

   False. The function $f(y)$ would have to have just two roots, and be positive to the right of each and negative to the left of each. Since there is an interval between the two roots, the function would have to be both positive and negative on this interval.