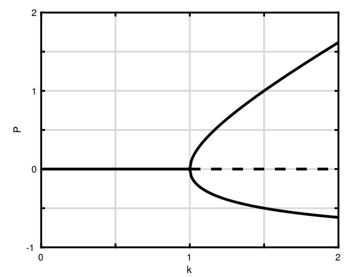
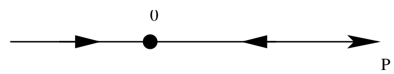
6. [16 points] Consider a animal population modeled by a differential equation $P' = f(P)$, where the function $f(P)$ involves a parameter k. At $k = 1$ there is a bifurcation point, as shown in the bifurcation diagram to the right. In this figure, solid curves indicate stable solutions while dashed curves indicate unstable ones. Even though $P < 0$ is not physically realizable, include negative values of P in your analysis in parts (a) and (b) below.

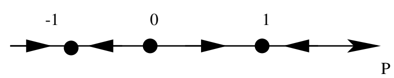


a. [6 points] Sketch phase diagrams for the differential equation $P' = f(P)$ for $k = 0.5$, $k = 1$ and $k = 1.5$.

Solution: We can draw the phase diagrams by using vertical slices of the bifurcation diagram with different values of k. For $k = 0.5$, there is one equilibrium point, $P = 0$, and it is stable, so solutions below and above this must increase and decrease, respectively:

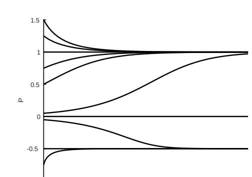


When $k = 1$ we are at the bifurcation point, but still have only the $P = 0$ solution, so the phase diagram is the same. When $k = 1.5$ there are three equilibrium solutions, at $P = -0.5$, $P = 0$ and $P = 1$. The first and last of these are stable, so we know that solutions move toward them. We therefore have the phase diagram shown below.



b. [6 points] Sketch qualitatively reasonable solution curves this equation for the case $k = 1.5$.

Solution: We know that there are equilibrium solutions at $P = -0.5$, $P = 0$ and $P = 1$. Near any of these the slope of the solution curve will be close to zero; further away, larger. This gives the figure shown to the right. Note that we are unable to say anything about the timescale on which the solutions evolve.



c. [4 points] Thinking of P as an animal population, what is the implication of the bifurcation point? Give a possible explanation for what k could measure.

Solution: The bifurcation point marks the point at which we expect the animals to be able to maintain a non-zero population. For $k \leq 1$, none of the population will survive; for $k > 1$ we expect to see a non-zero stable population. The parameter k could, therefore, model some aspect of the environment related to the carrying capacity. (Other explanations are possible.)