1. [15 points] Find real-valued solutions to each of the following, as indicated. If possible, find an explicit expression for y. *(Note that minimal partial credit will be given on this problem.)*

a. [5 points] Find the general solution to $y' + 5y = 3e^{6t}$

Solution: This is first-order and linear. An integrating factor is $\mu(t) = e^{\int 5\,dt} = e^{5t}$, so $(e^{5t}\,y)' = 3e^{11t}$. Integrating both sides, we have $e^{5t}\,y = \frac{3}{11}\,e^{11t} + C$, so that

$$y = \frac{3}{11}\,e^{6t} + C\,e^{-5t}.$$

b. [5 points] Find the solution to $(t+1)y' + y = 3$, $y(0) = 2$.

Solution: Note that this is equivalent to $y' + \frac{1}{t+1}\,y = \frac{3}{t+1}$ and to $y' = -\frac{1}{t+1}(y-3)$, so it is both linear and separable. An integrating factor is $\mu(t) = e^{\int (t+1)^{-1}\,dt} = e^{\ln(t+1)} = t+1$, so, multiplying both sides by μ, we have $((t+1)y)' = 3$. Integrating, $(t+1)y = 3t + C$, so that $y = \frac{3t}{t+1} + \frac{C}{t+1}$. Then, requiring that $y(0) = 2$, we have $C = 2$, and

$$y = \frac{3t}{t+1} + \frac{2}{t+1} = \frac{3t+2}{t+1}.$$

We could also solve this by separating: we have $\frac{y'}{y-3} = -\frac{1}{t+1}$, so that $\ln|y-3| = -\ln|t+1| + c$. Exponentiating both sides and taking $C = \pm e^c$, we have $y - 3 = \frac{C}{t+1}$, so that $y = 3 + \frac{C}{t+1}$. With $y(0) = 2$, $C = -1$, so that $y = 3 - \frac{1}{t+1} = \frac{3t+3-1}{t+1} = \frac{3t+2}{t+1}$, as before.

c. [5 points] Find the general solution to $y' + y^2 = ty^2$.

Solution: This is nonlinear, but separable. Separating variables, we have $y^{-2}y' = t - 1$, so that, integrating, $-y^{-1} = \frac{1}{2}t^2 - t + C$, and

$$y = -\frac{1}{\frac{1}{2}t^2 - t + C} = \frac{2}{k + 2t - t^2}.$$

(Where $k = -2C$.)