2. [14 points] Find real-valued solutions to each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] The general solution to $x' = x + 8y$, $y' = 2x + y$.

Solution: With $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ this is $\mathbf{x}' = \begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix} \mathbf{x}$. We then look for $\mathbf{x} = \mathbf{v}e^{\lambda t}$, so that

$$\det\left(\begin{pmatrix} 1-\lambda & 8 \\ 2 & 1-\lambda \end{pmatrix}\right) = (1-\lambda)^2 - 16 = (\lambda - 1)^2 - 16 = 0.$$

Thus $\lambda - 1 = \pm 4$, and $\lambda = -3, 5$. If $\lambda = -3$, we have $\begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. If $\lambda = 5$, $\begin{pmatrix} -4 & 8 \\ 2 & -4 \end{pmatrix} \mathbf{v} = \mathbf{0}$, and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Thus

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}.$$

b. [7 points] The solution to $\mathbf{x}' = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} -6 \\ 0 \end{pmatrix}$.

Solution: Again looking for $\mathbf{x} = \mathbf{v}e^{\lambda t}$, we have

$$\det\left(\begin{pmatrix} -\lambda & 4 \\ -1 & -\lambda \end{pmatrix}\right) = \lambda^2 + 4 = 0,$$

so that $\lambda = \pm 2i$. If $\lambda = 2i$, we have $\begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 2 \\ i \end{pmatrix}$. A complex-valued solution is

$$\mathbf{x} = \begin{pmatrix} 2 \\ i \end{pmatrix} (\cos(2t) + i\sin(2t) = \begin{pmatrix} 2\cos(2t) \\ -\sin(2t) \end{pmatrix} + i \begin{pmatrix} 2\sin(2t) \\ \cos(2t) \end{pmatrix} = \mathbf{a} + i\mathbf{b},$$

so a real-valued general solution is $\mathbf{x} = c_1\mathbf{a} + c_2\mathbf{b}$. Applying the initial condition, $c_1 = -3$ and $c_2 = 0$, so that

$$\mathbf{x} = \begin{pmatrix} -6\cos(2t) \\ 3\sin(2t) \end{pmatrix}.$$