3. [14 points] Lake Michigan has a volume of about 4,900 km^3 of water. Each year about 158 km^3 of that flows out to Lake Huron, and we may assume that an equal amount of water flows in from the rivers feeding the lake, rainfall and snowmelt. (Of course, the loss should really take into account evaporation as well, but ignore that here.)

a. [4 points] Write a differential equation modeling the amount $p(t)$ of a pollutant in the lake, assuming that the pollutant is added at a constant rate I_0 per year.

Solution: We have $p' = (\text{rate in}) - (\text{rate out})$. The rate in is given to be I_0, and, assuming that the lake water is well mixed, the rate out will be (pollutant concentration)(158) $= \frac{p}{4900}(158)$. Thus we have

$$p' = I_0 - \frac{158}{4900}\,p, \quad \text{or} \quad p' + \frac{158}{4900}\,p = I_0.$$

b. [6 points] For this and part **(c)** suppose that the equation that you obtained in **(a)** is $p' + \frac{1}{20}\,p = I_0$, and that the rate at which pollutant is added changes at at $t = 4$ as regulations on allowed pollution released are loosened. Thus, instead of a constant I_0, we have $I_0(t) = \begin{cases} 100, & t < 4 \\ 1000, & t \geq 4 \end{cases}$. Find $p(t)$ if $p(0) = 500$. You need not simplify any constants in your answer.

Solution: For $t < 4$, we solve with the integrating factor $\mu = e^{t/20}$: $(e^{t/20}p)' = 100e^{t/20}$, so $p = 2000 + Ce^{-t/20}$. The initial condition requires that $C = 500 - 2000 = -1500$, so $p = 2000 - 1500e^{-t/20}$. Then, for $t \geq 4$, we have $p' + \frac{1}{20}\,p = 1000$. Proceeding as before, $p = 20,000 + Ce^{-t/20}$, with initial condition $p(4) = 2000 - 1500e^{-1/5}$. Thus $2000 - 1500e^{-1/5} = 20,000 + Ce^{-1/5}$, and $C = -18,000e^{1/5} - 1500$. The solution to the problem is therefore

$$p(t) = \begin{cases} 2,000 - 1,500e^{-t/20} & t < 4 \\ 20,000 - (18,000e^{1/5} + 1,500)e^{-t/20} & t \geq 4. \end{cases}$$

c. [4 points] For the initial value problem you solved in **(b)**, on what domain does the solution exist, and where is it unique? On what domain would we expect a unique solution given our existence and uniqueness theorem? Is our result here consistent with the theorem?

Solution: We've found a function that is continuous for all t and unique, but it has a discontinuous derivative at $t = 4$. For it to be a solution to the differential equation, it must be differentiable (we've defined a solution to be a differentiable function). Therefore, we have a unique solution on $0 \leq t < 4$. The existence and uniqueness theorem guarantees a unique solution on the interval where the forcing and coefficient functions in the differential equation are continuous, which is $0 \leq t < 4$. Our result is therefore consistent with that (as it, of course, must be).