4. [14 points] Consider a population \( P \) that is modeled by the first-order differential equation

\[ P' = f(P). \]

In this problem we consider only \( P \geq 0 \), as a negative population is not physically relevant.

a. [4 points] If the phase line for the population is shown to the right, what could the differential equation be? Why?

Solution: There are many possible solutions to this; we need the function \( f(P) \) to have zeros at \( P = 0, P = 1, \) and \( P = 3 \), and to be negative for \( 0 < P < 1 \) and \( P > 1 \) and positive for \( 1 < P < 3 \). One such function is \( f(P) = -P(P-1)(P-3) \), so that \( P' = -P(P-1)(P-3) \).

b. [6 points] Now suppose that \( f(P) \) depends on a parameter \( H \), which measures the amount of harvesting of the population (e.g., if the population was fish, \( H \) could measure how many of the fish are caught through fishing). If the phase lines for \( H = 2 \), \( H = 4 \), and \( H = 6 \) are shown to the right, which, if any, of the following equations could model the population? Explain.

\[
\begin{align*}
\text{i. } & P' = -P(P-1)(P-H) \\
\text{ii. } & P' = P^3 - 4P^2 + HP \\
\text{iii. } & P' = -P(P^2 - HP + 4) \\
\text{iv. } & P' = -P(P^2 - 4P + H)
\end{align*}
\]

Solution: We must have equilibrium solutions as shown, and the derivative must have the appropriate sign to give the indicated phase lines. In particular, for large \( P \) we must have \( P' < 0 \): this disqualifies (ii). Then, when \( H = 4 \) the roots of the expression on the right-hand side of the equation must be \( P = 0, 2 \): this disqualifies (i). Finally, note that (iii) has roots \( P = 0 \) and \( P = \frac{1}{2}H \pm \frac{1}{2}\sqrt{H^2 - 16} \). When \( H = 6 \) this has a positive root, which shouldn’t be the case, so it is also not correct. By elimination, the equation must be (iv); this has roots \( P = 0 \) and \( P = 2 \pm \sqrt{4 - H} \), which gives exactly the phase lines shown. This is therefore correct.

c. [4 points] Finally, sketch a qualitatively accurate plot of solutions to the equation for the case \( H = 4 \).

Solution: An appropriate sketch is something like the following.