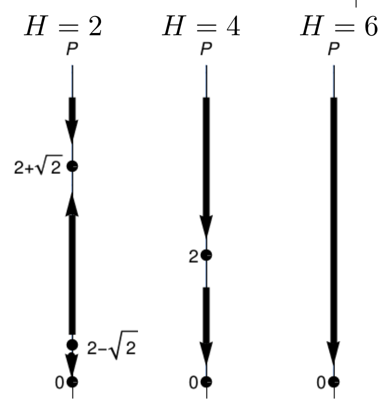
4. [14 points] Consider a population P that is modeled by the first-order differential equation $P' = f(P)$. In this problem we consider only $P \geq 0$, as a negative population is not physically relevant.

a. [4 points] If the phase line for the population is shown to the right, what could the differential equation be? Why?

Solution: There are many possible solutions to this; we need the function $f(P)$ to have zeros at $P = 0$, $P = 1$, and $P = 3$, and to be negative for $0 < P < 1$ and $P > 1$ and positive for $1 < P < 3$. One such function is $f(P) = -P(P-1)(P-3)$, so that $P' = -P(P-1)(P-3)$.

b. [6 points] Now suppose that $f(P)$ depends on a parameter H, which measures the amount of harvesting of the population (e.g., if the population was fish, H could measure how many of the fish are caught through fishing). If the phase lines for $H = 2$, $H = 4$, and $H = 6$ are shown to the right, which, if any, of the following equations could model the population? Explain.

i. $P' = -P(P-1)(P-H)$ ii. $P' = P^3 - 4P^2 + HP$
iii. $P' = -P(P^2 - HP + 4)$ iv. $P' = -P(P^2 - 4P + H)$



Solution: We must have equilibrium solutions as shown, and the derivative must have the appropriate sign to give the indicated phase lines. In particular, for large P we must have $P' < 0$: this disqualifies (ii). Then, when $H = 4$ the roots of the expression on the right-hand side of the equation must be $P = 0, 2$: this disqualifies (i). Finally, note that (iii) has roots $P = 0$ and $P = \frac{1}{2}H \pm \frac{1}{2}\sqrt{H^2 - 16}$. When $H = 6$ this has a positive root, which shouldn't be the case, so it is also not correct. By elimination, the equation must be (iv); this has roots $P = 0$ and $P = 2 \pm \sqrt{4 - H}$, which gives exactly the phase lines shown. This is therefore correct.

c. [4 points] Finally, sketch a qualitatively accurate plot of solutions to the equation for the case $H = 4$.

Solution: An appropriate sketch is something like the following.