**6**. [12 points] Identify each of the following as true or false, by circling "True" or "False" as appropriate, and provide a short (one or two sentence) explanation indicating why you selected that answer.

**a**. [3 points] The initial value problem $(y^2 - 1)y' = (t - 1)$, $y(0) = 0$, is guaranteed to have a unique solution for all times $t > 0$.

True **[False]**

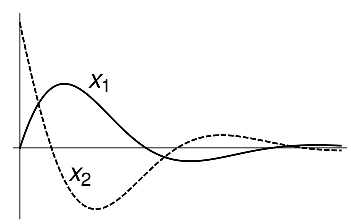
*Solution:* We note that this is a nonlinear equation of the from $y' = f(t, y)$, with $f$ and $\frac{\partial f}{\partial y}$ being discontinuous only at $y = \pm 1$. Thus we are guaranteed a unique solution through the initial condition, but the interval on which it exists may be constrained. In this case, it exists only for $t < 1 + \sqrt{\frac{7}{3}}$ (though this isn't immediately obvious from the equation).

**b**. [3 points] If the eigenvalues of a $2 \times 2$ constant, real-valued matrix $\mathbf{A}$ are $\lambda_1 = 0$ and $\lambda_2 = 1$, then the system of algebraic equations $\mathbf{Ax} = \mathbf{0}$ has infinitely many nonzero solutions.

**[True]** False

*Solution:* If $\lambda = 0$ is an eigenvalue, then we know that $\det(\mathbf{A} - 0\mathbf{I}) = \det(\mathbf{A}) = 0$. This means that $\mathbf{Ax} = \mathbf{b}$ has no or an infinite number of solutions for any $\mathbf{b}$; if $\mathbf{b} = \mathbf{0}$, there must be an infinite number, of which the zero solution is one.

**c**. [3 points] If $\mathbf{A} = \begin{pmatrix} -1 & a \\ -a & -1 \end{pmatrix}$, then component plots for the system of equations $\mathbf{x}' = \mathbf{Ax}$ will appear as in the figure to the right for all real values of $a$.




**[True]** False

*Solution:* Technically this is false, but if one is not being too tricky and assumes that $a \neq 0$, it is true. In that case eigenvalues of $\mathbf{A}$ are given by $(\lambda+1)^2 = -a^2$, so $\lambda = -1 \pm ia$, and solutions will be decaying and oscillatory, which is what is shown here. If $a = 0$, however, we have $\lambda = -1$, twice, and there are two linearly independent eigenvectors. Thus in that case we would have strictly decaying solutions. (Either of these responses were accepted as correct for this problem.)

**d**. [3 points] A first-order problem such as $y' = t\sin(y) + \cos(y)$, which is neither linear nor separable, is amenable to qualitative analysis by drawing a phase line and sketching qualitatively accurate solution curves.

True **[False]**

*Solution:* This type of qualitative analysis only works with autonomous equations, for which the dependence on the independent variable $t$ is implicit.