7. [16 points] The van der Pol equation has the form $x'' + \mu \frac{df}{dx} x' + x = 0$. In this problem suppose that $f(x) = -\sin(x)$, so that the equation becomes $x'' - \mu\cos(x)x' + x = 0$.

a. [4 points] Letting $x_1 = x$ and $x_2 = x'$, write this as a system of two first-order differential equations in x_1 and x_2.

Solution: We have

$$x_1' = x_2$$
$$x_2' = -x_1 + \mu\cos(x_1)x_2.$$

b. [4 points] Use a Taylor expansion to linearize the original equation at the critical point $x = 0$.

Solution: Using the Taylor series for $\cos(x)$, we have $\cos(x) = 1 - \frac{1}{2}x^2 + \cdots$, so that the equation is $x'' - \mu(1 - \frac{1}{2}x^2 + \cdots)x' + x = 0$. Expanding and dropping nonlinear terms, we have

$$x'' - \mu x' + x = 0.$$

Problem 7, continued.

c. [4 points] Suppose that the equation you obtained in **(b)** is, for some value of μ,

$$x'' + 3x' + 2x = 0.$$

Write this as a matrix equation in $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and solve it.

Solution: We have

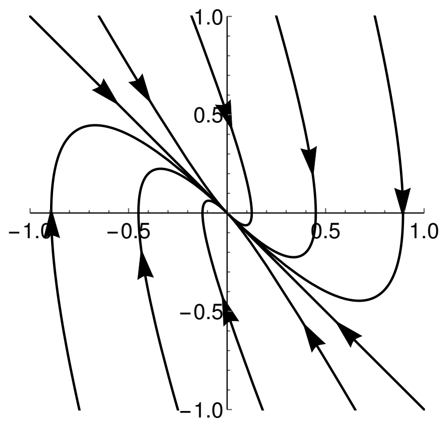
$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 3x_2 \end{aligned}, \quad \text{or} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Letting $\mathbf{x} = \mathbf{v}e^{\lambda t}$, we must have $\det(\mathbf{A} - \lambda \mathbf{I}) = (-\lambda)(-\lambda - 3) + 2 = \lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1) = 0$. Thus $\lambda = -2$ and $\lambda = -1$. The eigenvector for $\lambda = -2$ satisfies $\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Similarly, for $\lambda = -1$, we have $\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The general solution to the problem is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}.$$

d. [4 points] Sketch a phase portrait given your solution in **(c)**. What does it tell us about the long-term behavior of the current x in the circuit?

Solution: The two straight-line solutions in the problem lie along $y = -x$ and $y = -x/2$, and the latter decays much faster than the former. Thus we have the phase portrait shown below.



This indicates that the current ($x = x_1$) will asymptotically approach 0 for all initial currents.