2. [15 points] Lake Huron and Lake Erie are two of the Great Lakes, as shown to the right. The volume of Lake Huron is very approximately 4,000 km^3, and that of Lake Erie approximately 500 km^3. We assume that the flow into and out of both lakes is the same, approximately 200 km^3/year, and that all water that flows out of Lake Huron flows into Lake Erie. Suppose that a ruptured oil line adds 30×10^9 kg/year of oil into Lake Huron.

a. [5 points] Write a system of equations for x_1, the amount of oil in Lake Huron, and x_2, the amount in Lake Erie. Assume that the oil is well mixed in either lake, and that the water entering Lake Huron is clean.

Solution: For both lakes we have $\frac{dx}{dt}$ = input − output. For Lake Huron the input is $r = 3 \times 10^{10}$ kg/year. The output for both lakes is (200 km^3/year)(concentration), where concentration = x/volume. The output from Lake Huron is the input for Lake Erie. Thus we have the system

$$\frac{dx_1}{dt} = r - \frac{200}{4000}\,x_1, \qquad \frac{dx_2}{dt} = \frac{200}{4000}\,x_1 - \frac{200}{500}\,x_2.$$

b. [5 points] Solve your equation for x_1 directly and use that to solve for x_2. It will likely be convenient to write your answer in terms of the constant $r = 3 \times 10^{10}$.

Solution: The equation for x_1 is $x_1' + 0.05x_1 = r$, so an integrating factor is $\mu = e^{0.05t}$. Multiplying by μ and integrating, we have $e^{0.05t}\,x_1 = 20re^{0.05t} + C$, so that $x_1 = 20r + Ce^{-0.05t}$. We assume that $x_1(0) = 0$, so that $x_1 = 20r(1 - e^{-0.05t})$.
Plugging this into the equation for x_2, we have $x_2' + 0.4x_2 = r(1 - e^{-0.05t})$. Proceeding similarly, an integrating factor is $\mu = e^{0.4t}$, so that $(e^{0.4t}\,x_2)' = r\,e^{0.4t} - r\,e^{0.35t}$. Integrating both sides, $e^{0.4t}\,x_2 = \frac{5}{2}\,r\,e^{0.4t} - \frac{20}{7}\,r\,e^{0.35t} + C$, so that $x_2 = \frac{5}{2}\,r - \frac{20}{7}\,r\,e^{-0.05t} + Ce^{-0.4t}$. With $x_2(0) = 0$, $C = (\frac{20}{7} - \frac{5}{2})r = \frac{5}{14}\,r$, and $x_2 = 5r(\frac{1}{2} - \frac{4}{7}\,e^{-0.05t} + \frac{1}{14}\,e^{-0.4t})$.

c. [5 points] Using your work in (a) and (b), write your system from (a) as a matrix equation, and write the solution as a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. What are the eigenvectors of the coefficient matrix in your system?

Solution: We have $\mathbf{x}' = \begin{pmatrix} -0.05 & 0 \\ 0.05 & -0.4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} r \\ 0 \end{pmatrix}$. The solution is $\mathbf{x} = \begin{pmatrix} -20r \\ -20r/7 \end{pmatrix} e^{-0.05t} + \begin{pmatrix} 0 \\ 5r/14 \end{pmatrix} e^{-0.4t} + \begin{pmatrix} 20r \\ 5r/2 \end{pmatrix}$, from which we see that the eigenvectors are $\begin{pmatrix} 7 \\ 20 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.