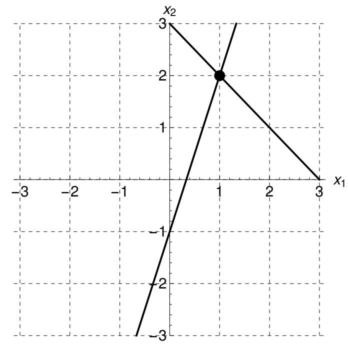
3. [14 points] In the following, the matrices $\mathbf{A}$ and $\mathbf{B}$ are 2×2 real-valued matrices. The vector $\mathbf{x}$ is a 2×1 vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

a. [7 points] If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and the solution to $\mathbf{Ax} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ is illustrated in the figure to the right, what are the eigenvalues of $\mathbf{A}$?



Solution: We see that the two lines shown are $x_2 = 3 - x_1$ and $x_2 = 3x_1 - 1$. Because the right-hand side of the system of equations given is $\begin{pmatrix} 3 & -1 \end{pmatrix}$, the equations must be $x_1 + x_2 = 3$ and $-3x_1 + x_2 = -1$. Therefore the rows of $\mathbf{A}$ must be $\begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -3 & 1 \end{pmatrix}$, and because the solution shown is $(1, 2)$, the matrix $\mathbf{A}$ must be $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$. The eigenvalues are determined by $(1 - \lambda)(1 - \lambda) + 3 = (\lambda - 1)^2 + 3 = 0$, so that $\lambda = 1 \pm i\sqrt{3}$.

b. [7 points] Suppose that $\mathbf{B}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\mathbf{B}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$. What is the general solution to $\mathbf{x}' = \mathbf{Bx}$?

Solution: These two systems tell us that the eigenvalues and eigenvectors of $\mathbf{B}$ are $\lambda = 3$ and $\lambda = -2$, with $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. The general solution is therefore $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$.