4. [15 points] Find explicit, real-valued solutions to the following, as indicated.

a. [8 points] $x' = x + 2y$, $y' = 3x - 4y$, $x(0) = 1$, $y(0) = 4$.

Solution: The coefficient matrix for this system is $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$, so that eigenvalues satisfy $(1-\lambda)(-4-\lambda) - 6 = \lambda^2 + 3\lambda - 10 = (\lambda+5)(\lambda-2) = 0$. Thus $\lambda = -5$ or $\lambda = 2$. If $\lambda = -5$, eigenvectors satisfy $6v_1 + 2v_2 = 0$, so that $\mathbf{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. If $\lambda = 2$, eigenvectors satisfy $-v_1 + 2v_2 = 0$, so that $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The general solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}.$$

Applying the initial conditions, we have $c_1 + 2c_2 = 1$, and $-3c_1 + c_2 = 4$. Adding three times the first to the second, we have $7c_2 = 7$, so $c_2 = 1$. Then $c_1 = 1 - 2 = -1$, so our solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-5t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}.$$

b. [7 points] $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -8 & -3 \end{pmatrix} \mathbf{x}$.

Solution: Eigenvalues of the coefficient matrix are given by $(1-\lambda)(-3-\lambda) + 8 = \lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 4 = 0$, so $\lambda = -1 \pm 2i$. With $\lambda = -1 + 2i$, the eigenvector satisfies $(2-2i)v_1 + v_2 = 0$, so we have $\mathbf{v} = \begin{pmatrix} 1 \\ -2+2i \end{pmatrix}$ (or any constant multiple thereof; in particular, $\begin{pmatrix} -1-i \\ 4 \end{pmatrix}$ is another option). To find a real-valued solution, we separate out the real and imaginary parts of $\mathbf{x} = \mathbf{v}e^{\lambda t}$. These are the real and imaginary parts of

$$\mathbf{x} = \begin{pmatrix} 1 \\ -2+2i \end{pmatrix} e^{-t}(\cos(2t) + i\sin(2t)),$$

which are $\mathrm{Re}(\mathbf{x}) = \begin{pmatrix} \cos(2t) \\ -2\cos(2t) - 2\sin(2t) \end{pmatrix} e^{-t}$ and $\mathrm{Im}(\mathbf{x}) = \begin{pmatrix} \sin(2t) \\ 2\cos(2t) - 2\sin(2t) \end{pmatrix} e^{-t}$. The general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} \cos(2t) \\ -2\cos(2t) - 2\sin(2t) \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \sin(2t) \\ 2\cos(2t) - 2\sin(2t) \end{pmatrix} e^{-t}.$$

(With $\mathbf{v} = \begin{pmatrix} -1-i \\ 4 \end{pmatrix}$, we get $\mathbf{x} = c_1 \begin{pmatrix} -\cos(2t) + \sin(2t) \\ 4\cos(2t) \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\cos(2t) - \sin(2t) \\ 4\sin(2t) \end{pmatrix} e^{-t}$.)