5. [15 points] Consider the initial value problem $y' = -\frac{1}{2}\,y + \sin(y)$, $y(t_0) = y_0$.

a. [5 points] Without trying to solve it, does this initial value problem have a solution? Does your answer depend on the values of t_0 and y_0? Explain.

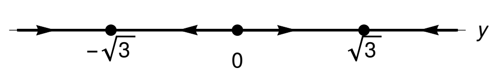
Solution: We see that $f(t, y) = -\frac{1}{2}\,y + \sin(y)$ is continuous and has a continuous first partial f_y (for all t and y). Therefore, our theorem on the existence and uniqueness of solutions guarantees that there is a unique solution through $y(t_0) = y_0$ for any t_0 and any y_0. However, we do not, a priori, know the full domain of this solution.

b. [5 points] By using the Taylor expansion for $\sin(y)$ near the critical point $y = 0$, write a linear equation approximating this equation and solve it. If we start with $y(0) = y_0$ with y_0 small, what does it predict will happen to the solution of the (nonlinear) problem? Is the critical point $y = 0$ stable or unstable?

Solution: We have $\sin(y) \approx y - \frac{1}{6}y^3 + \frac{1}{120}y^5 - \cdots$, so our linear approximation is $y' = \frac{1}{2}\,y$. The solution is tremendously easy: $y = y_0 e^{t/2}$, and solutions grow. Thus, this predicts that for the nonlinear problem, if we start with y_0 close to zero solution trajectories will grow away from $y = y_0$. The critical point is unstable.

c. [5 points] Retain another term in the expansion for $\sin(y)$ and write a new differential equation that approximates the equation we started with. Find all critical points, draw a phase line, and explain what it predicts for the behavior of the system for large times.

Solution: If we retain another term, we have $y' = \frac{1}{2}\,y - \frac{1}{6}\,y^3$. Critical points are given by $0 = y(1 - \frac{1}{3}\,y^2)$, which are when $y = 0$ and $y = \pm\sqrt{3}$. With $f(y) = \frac{1}{2}\,y - \frac{1}{6}\,y^3$, we have $f'(y) = \frac{1}{2} - \frac{1}{3}\,y^2$, so $f'(0) > 0$ and $f'(\pm\sqrt{3}) < 0$, indicating that $y = 0$ is unstable and $y = \pm\sqrt{3}$ are asymptotically stable. This is indicated in the phase line shown below.



This suggests that every positive initial condition will eventually converge to $y = \sqrt{3}$, and every negative initial condition will converge to $y = -\sqrt{3}$. (And, of course, an initial condition of $y = 0$ will remain there.)