5. [15 points] Consider the initial value problem \( y' = -\frac{1}{2} y + \sin(y), \ y(t_0) = y_0. \)

a. [5 points] Without trying to solve it, does this initial value problem have a solution? Does your answer depend on the values of \( t_0 \) and \( y_0 \)? Explain.

**Solution:** We see that \( f(t, y) = -\frac{1}{2} y + \sin(y) \) is continuous and has a continuous first partial \( f_y \) (for all \( t \) and \( y \)). Therefore, our theorem on the existence and uniqueness of solutions guarantees that there is a unique solution through \( y(t_0) = y_0 \) for any \( t_0 \) and any \( y_0 \). However, we do not, a priori, know the full domain of this solution.

b. [5 points] By using the Taylor expansion for \( \sin(y) \) near the critical point \( y = 0 \), write a linear equation approximating this equation and solve it. If we start with \( y(0) = y_0 \) with \( y_0 \) small, what does it predict will happen to the solution of the (nonlinear) problem? Is the critical point \( y = 0 \) stable or unstable?

**Solution:** We have \( \sin(y) \approx y - \frac{1}{6} y^3 + \frac{1}{120} y^5 - \cdots \), so our linear approximation is \( y' = \frac{1}{2} y \). The solution is tremendously easy: \( y = y_0 e^{t/2} \), and solutions grow. Thus, this predicts that for the nonlinear problem, if we start with \( y_0 \) close to zero solution trajectories will grow away from \( y = y_0 \). The critical point is unstable.

c. [5 points] Retain another term in the expansion for \( \sin(y) \) and write a new differential equation that approximates the equation we started with. Find all critical points, draw a phase line, and explain what it predicts for the behavior of the system for large times.

**Solution:** If we retain another term, we have \( y' = \frac{1}{2} y - \frac{1}{6} y^3 \). Critical points are given by \( 0 = y(1 - \frac{1}{3} y^2) \), which are when \( y = 0 \) and \( y = \pm \sqrt{3} \). With \( f(y) = \frac{1}{2} y - \frac{1}{6} y^3 \), we have \( f'(y) = \frac{1}{2} - \frac{1}{3} y^2 \), so \( f'(0) > 0 \) and \( f'(|\pm \sqrt{3}|) < 0 \), indicating that \( y = 0 \) is unstable and \( y = \pm \sqrt{3} \) are asymptotically stable. This is indicated in the phase line shown below.

This suggests that every positive initial condition will eventually converge to \( y = \sqrt{3} \), and every negative initial condition will converge to \( y = -\sqrt{3} \). (And, of course, an initial condition of \( y = 0 \) will remain there.)