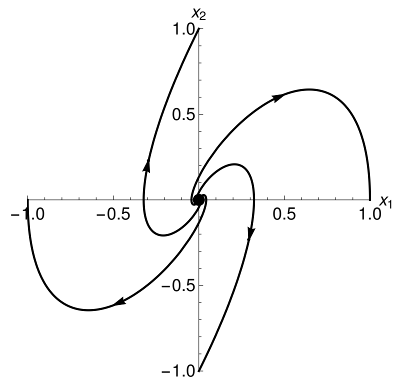
6. [14 points] Suppose that the phase portrait to the right is the phase portrait for a system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 constant, real-valued matrix. If the system is obtained by rewriting a second order equation as a system of first-order equations, give a possible matrix for $\mathbf{A}$. Explain how you know your choice is correct.



Solution: Because the system is a rewritten second order equation, we must have started with an equation $x'' - bx' - ax = 0$; taking $x' = y$, the resulting system is $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. We know that the eigenvalues of the coefficient matrix must be complex-valued (because the phase portrait shows a spiral) and have a positive real part (because trajectories move out from the origin). The eigenvalues of our matrix are given by $-\lambda(b-\lambda) - a = \lambda^2 - b\lambda - a = 0$ For convenience, let $b = 2$. Then the characteristic equation is $\lambda^2 - 2\lambda - a = (\lambda - 1)^2 - (a + 1) = 0$, and if $a < 1$ we have complex roots. For example, if $a = -2$, we have $\lambda = 1 \pm i$. Our matrix could therefore be

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}.$$

(Note that at $(1, 0)$, this gives $\mathbf{x}' = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$, a trajectory moving straight down past the x_1 axis, as shown in the figure.)

More generally, because $\lambda^2 - b\lambda - a = 0$ we have $\lambda = \frac{1}{2}\,b \pm \frac{1}{2}\sqrt{b^2 + 4a}$, and therefore we must have $b > 0$ and $b^2 + 4a < 0$, so that $a < -\frac{1}{4}\,b^2$. Any matrix satisfying these conditions will give us the desired phase portrait.