7. [12 points] Suppose that the matrix $\mathbf{A}$ has eigenvalues $\lambda = -1$ and $\lambda = -2$, with corresponding eigenvectors $\mathbf{v}_{-1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_{-2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. If the solution to $\mathbf{Ax} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ is $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, sketch the phase portrait for the system $\mathbf{x}' = \mathbf{Ax} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$. Explain how you get your answer.

Solution: First note that if $\mathbf{A}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, then $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is the equilibrium solution of $\mathbf{x}' = \mathbf{Ax} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$. Thus if we let $\mathbf{x} = \mathbf{x}_0 + \mathbf{u}$, then $\mathbf{u}' = \mathbf{Au}$.

The phase portrait for the system $\mathbf{u}' = \mathbf{Au}$ is obtained as follows. There are two negative real eigenvalues, so along lines through the origin given by the two eigevectors all trajectories move to the origin. Away from these, trajectories will collapse fastest in the direction of the vector $\mathbf{v}_{-2}$, and then approach the origin along $\mathbf{v}_{-1}$.

Then to get from $\mathbf{u}$ to $\mathbf{x}$, we add $\mathbf{x}_0$. We therefore obtain the phase portrait shown below.