2. [15 points] Solve each, finding explicit real-valued solutions as indicated.

a. [8 points] Solve the initial value problem $x' = -y$, $y' = 12x - 7y$, $x(0) = 2$, $y(0) = 1$.

Solution: In matrix form, this is $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 12 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. The eigenvalues of the coefficient matrix are given by $\det(\begin{pmatrix} -\lambda & -1 \\ 12 & -7-\lambda \end{pmatrix}) = \lambda^2 + 7\lambda + 12 = (\lambda+3)(\lambda+4) = 0$. Thus $\lambda = -4$ or $\lambda = -3$. Note that the first row of the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ for the eigenvector gives $\mathbf{v} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$, so the corresponding eigenvectors are $\mathbf{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. The general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t}.$$

Applying the initial conditions, we have $c_1 + c_2 = 2$ and $4c_1 + 3c_2 = 1$. Subtracting the second from four times the first, $c_2 = 7$, so that $c_1 = -5$. The solution is

$$\mathbf{x} = -5 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-4t} + 7 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t}.$$

b. [7 points] Find the general solution to $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 6 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Solution: The eigenvalues of the coefficient matrix are given by $(6-\lambda)(-2-\lambda) + 20 = \lambda^2 - 4\lambda + 8 = (\lambda - 2)^2 + 4 = 0$. Thus $\lambda = 2 \pm 2i$. If $\lambda = 2 + 2i$, the components of the eigenvector satisfy $(4-2i)v_1 - 5v_2 = 0$, so we may take $\mathbf{v} = \begin{pmatrix} 5 \\ 4-2i \end{pmatrix}$. A complex-valued solution is therefore $\mathbf{x} = \begin{pmatrix} 5 \\ 4-2i \end{pmatrix} e^{2t}(\cos(2t) + i\sin(2t))$. Separating the real and imaginary parts of this, we have

$$\mathbf{x} = c_1 \begin{pmatrix} 5\cos(2t) \\ 4\cos(2t) + 2\sin(2t) \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 5\sin(2t) \\ -2\cos(2t) + 4\sin(2t) \end{pmatrix} e^{2t}.$$

Alternately, we could take $\mathbf{v} = \begin{pmatrix} 2+i \\ 2 \end{pmatrix}$, so that

$$\mathbf{x} = c_1 \begin{pmatrix} 2\cos(2t) - \sin(2t) \\ 2\cos(2t) \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} \cos(2t) + 2\sin(2t) \\ 2\sin(2t) \end{pmatrix} e^{2t}.$$