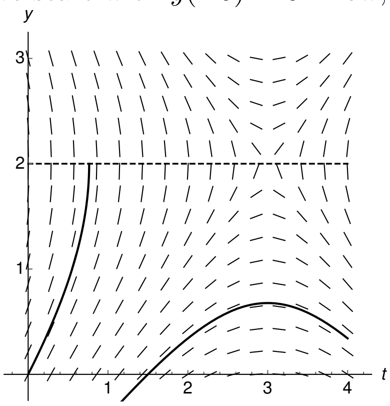
3. [12 points] Suppose we are solving the initial value problem $y' = \dfrac{t-3}{y-2}$, $y(0) = y_0$.

a. [6 points] A direction field for the differential equation is shown to the right, below. Using this and your knowledge of the differential equation, explain what the solution will look like if we start with the initial condition $y(0) = 0$, and if we start with $y(1.5) = 0$. How, and why, are these solutions different?

(The printed exam copy had $y(1) = 0$ for the second initial condition. This was supposed to be $y(1.5)$; through $(0, 1)$ the solution is $y = t - 1$.)



Solution: With $y(0) = 0$, we expect the solution that bends up from $(0, 0)$ until it gets to $y = 2$. At $y = 2$, the right-hand side of the differential equation becomes undefined and we expect that we may have trouble continuing the solution. In this case it appears that the solution tries to bend back on itself, which it cannot do. Therefore, we expect that at $y = 2$ we expect the solution to end. This makes sense, because we would anticipate that any initial condition $y(t_0) = 2$ may not have a solution, because of the existence and uniqueness theorem.

From $y(1.5) = 0$, the solution appears to grow and turn over, then continuing to larger negative values. Thus y never reaches $y = 2$, and we expect the solution to exist for all times.

b. [6 points] Solve the problem with initial condition $y(0) = 0$. Based on your solution, for what values of t and y does your solution exist? How is this related to the existence and uniqueness theorem?

Solution: Separating values and integrating, we have $\frac{1}{2}y^2 - 2y = \frac{1}{2}t^2 - 3t + C$, and the initial condition $y(0) = 0$ requires that $C = 0$. We can find an explicit solution for y by multiplying by 2 and using the quadratic formula: $y^2 - 4y - t^2 + 6t = 0$, so $y = 2 \pm \sqrt{4 - (-t^2 + 6t)}$. To have $y(0) = 0$ we take the negative, so $y = 2 - \sqrt{t^2 - 6t + 4}$. This will work until $y = 0$, which is when $t^2 - 6t + 4 = (t-3)^2 - 5 = 0$, so $t = 3 \pm \sqrt{5}$. Thus we expect the solution to exist for $0 \le t < 3 - \sqrt{5}$, $0 \le y < 2$.

At $y = 2$, the right-hand side of the equation become discontinuous, and the existence and uniqueness theorem doesn't guarantee a solution through the initial condition $y(3 - \sqrt{5}) = 2$. Thus we may expect the solution to break down there.