4. [15 points] Consider the system of differential equations given by $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ with the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

a. [4 points] If $\mathbf{P}(t) = \begin{pmatrix} 0 & 1 \\ -2t^{-2} & 2t^{-1} \end{pmatrix}$, is this a linear or nonlinear problem? If we apply the initial condition, will there be a unique solution? Explain.

Solution: This is a linear problem, though non-constant coefficient. Accordingly, there will be a unique solution through any $\mathbf{x}(t_0) = \mathbf{x}_0$ where $\mathbf{P}(t)$ is continuous. That is, through any $t_0 \neq 0$. The solution will exist on the interval $(0, \infty)$ or $(-\infty, 0)$, depending on whether $t_0 > 0$ or $t_0 < 0$.

b. [6 points] If $\mathbf{P}(t) = \mathbf{A}$, a 2×2 constant real-valued matrix, and if a general solution to the system is $\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda t} + c_2(t\mathbf{v}_1 + \mathbf{v}_2)e^{\lambda t}$, how many solutions are there to each of the following algebraic systems of equations? Why?
(i) $\mathbf{A}\mathbf{x} = 2\lambda\mathbf{x}$
(ii) $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{v}_1$

Solution: (i) Note that we know the only eigenvalue is λ, with eigenvector $\mathbf{v}_1$. Thus 2λ is not an eigenvalue, and we cannot find a non-zero solution to $\mathbf{A}\mathbf{x} = 2\lambda\mathbf{x}$. The only solution is $\mathbf{x} = \mathbf{0}$.
(ii) In this case, we know there are an infinite number of solutions: we're solving for the generalized eigenvector, which is only unique up to an additive multiple of the eigenvector $\mathbf{v}_1$. We can see this directly by noting that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = 0$: thus

$$(\mathbf{A} - \lambda\mathbf{I})(\mathbf{v}_2 + k\mathbf{v}_1) = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 + k(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{v}_1 + \mathbf{0}.$$

c. [5 points] If $\mathbf{P}(t) = \mathbf{B}$, a 2×2 constant real-valued matrix, and a solution to the system is $\mathbf{x} = \begin{pmatrix} \cos(3t) \\ \cos(3t) - 2\sin(3t) \end{pmatrix} e^{-4t}$, what are the eigenvalues and eigenvectors of $\mathbf{B}$?

Solution: If this is a solution, we can immediately see that the eigenvalues must be $\lambda = -4 \pm 3i$, because the time dependence of the solution comes from $e^{\lambda t} = e^{(\mu+\nu)t} = e^{\mu t}(\cos(\nu t) + i\sin(\nu t))$. We may then guess that $\mathbf{x}$ is the real or imaginary part of $\mathbf{v}e^{\lambda t}$, where $\mathbf{v}$ is the corresponding eigenvector. This leads us to conclude that $\mathbf{v} = \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix}$, if $\mathbf{x}$ is the real part, or $\mathbf{v} = \begin{pmatrix} i \\ -2 + i \end{pmatrix}$, if $\mathbf{x}$ is the imaginary part.