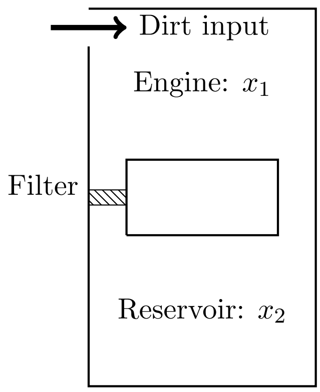
5. [16 points] In internal combustion engines, oil is circulated from a reservoir, around moving parts to lubricate them, and back to the reservoir. As it circulates, it collects dirt from the engine. To remove the dirt, oil from the reservoir is passed through a filter. A simple model for this system is shown to the right. Dirt is "added" to the oil in the engine, and we denote the amount of dirt in the engine compartment as x_1 and that in the reservoir as x_2. Suppose that the amount of oil in the engine compartment is 3 quarts and in the reservoir there are 2 quarts. Oil moves from the engine to the reservoir and back at a rate of 1 quart/minute, and the filter removes a fraction of the dirt from the oil returning from the reservoir to the engine.



a. [4 points] Suppose that x_1 and x_2 are measured in grams. A model for the amount of dirt in either compartment is

$$x_1' = -\frac{1}{3}\,x_1 + \frac{3}{5}\left(\frac{1}{2}\right)x_2 + 3, \quad x_2' = \frac{1}{3}\,x_1 - \frac{1}{2}\,x_2.$$

How much dirt is added to the oil in the engine? Why is there the term $\frac{1}{3}\,x_1$ in each equation, and why does it have this form? How much of the dirt in the oil is removed by the filter, and how do you know?

Solution: The addition term is the +3 in the equation for x_1', as that's the one place we have a constant addition of dirt. Thus we're adding an impressive 3 g/min of dirt.

The term $\frac{1}{3}x_1 = (1\text{ quart/min})(\frac{x_1}{3}\text{g/quart})$ is the rate at which dirt is moved from the engine (at 1 quart/min, with a concentration of $\frac{x_1}{3}$ g/quart) to the reservoir.

Finally, the filter removes 40% of the dirt: the term $\frac{1}{2}x_2$ in the last equation represents the dirt removed from the reservoir, of which $\frac{3}{5} = 60\%$ arrives at the engine.

b. [4 points] Find the equilibrium solution(s) for this system. What is the physical meaning of the equilibrium solution?

Solution: Equilibrium solutions are constant, so we have

$$0 = -\frac{1}{3}x_1 + \frac{3}{10}x_2 + 3, \quad 0 = \frac{1}{3}x_1 - \frac{1}{2}x_2.$$

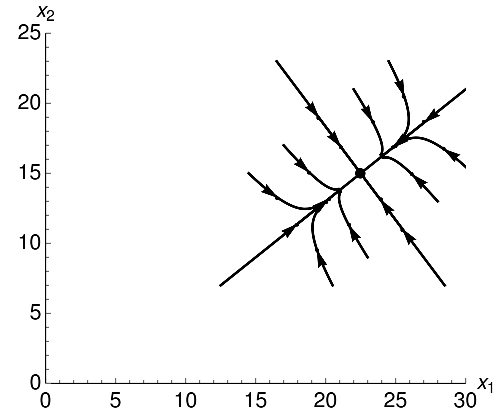
Adding the equations, we have $0 = -\frac{2}{10}x_2 + 3$, so that $x_2 = 15$. Then $x_1 = \frac{3}{2}x_2 = \frac{45}{2} = 22.5$. These are the long-term amounts of dirt that we expect to find in the engine and reservoir.

Problem 5, continued. We are considering the system

$$x_1' = -\frac{1}{3}x_1 + \frac{3}{5}\left(\frac{1}{2}\right)x_2 + 3, \quad x_2' = \frac{1}{3}x_1 - \frac{1}{2}x_2.$$

c. [4 points] The eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} -1/3 & 3/10 \\ 1/3 & -1/2 \end{pmatrix}$ are, approximately, $\lambda_1 = -0.75$ and $\lambda_2 = -0.1$, with $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$. Sketch a phase portrait for this system.

Solution: Note that we expect the phase portrait for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, shifted to the equilibrium point $(22.5, 15)$. The phase portrait is a stable node, as shown in the figure below.



d. [4 points] Suppose that, somehow, we start with the initial condition $x_1(0) = 22.5$ and $x_2(0) = 0$. Use your work in (b) to sketch, approximately, what you expect x_1 and x_2 to look like as functions of time.

Solution: From the phase portrait, above, we see that x_2 will increase, asymptotically approaching $x_2 = 15$, and x_1 will initially decrease slightly, then increase to $x_1 = 22.5$. This gives the component curves shown below, with x_1 solid and x_2 dashed, and the equlibria shown dotted.