6. [15 points] In lab 1 we considered the Gompertz equation, \( y' = ry \ln(K/y) \). We explore this further in this problem.

a. [5 points] Consider the initial condition \( y(0) = 1 \). Find a linear approximation to the Gompertz equation that is valid near this initial condition. Under what conditions would you expect your approximation to be accurate?

Solution: Noting that \( y' = ry(\ln(K) - \ln(y)) \), we can use the Taylor expansion of \( \ln(y) = 0 + (y - 1) + \cdots \) to linearize the equation. To retain only linear terms, we truncate the log at the constant term 0, and so have \( y' = r \ln(K) \). This is a reasonable approximation, though a little bit of a fudge as we haven’t expanded the \( y \) term in the equation.

If we are slightly more careful, we also expand the linear term \( y \) as \( y = 1 + (y - 1) \). Then \( y' = r(1 + (y - 1))(\ln(K) - (0 + (y - 1))) \). To retain only terms in \((y - 1)^0\) or \((y - 1)^1\) we must truncate the expansion of the logarithm at the constant (0) term, so that \( y' = r(1 + (y - 1))(\ln(K)) = r \ln(K) \). This retains a constant term \( r \) because \( y = 1 \) isn’t a critical point of the equation.

In any case, this is valid when \( y \) is near 1, and as \( y \) moves away from that we would expect the approximation to rapidly get worse.

b. [5 points] We found that for \( y \) near \( K \), the Gompertz equation is approximated as \( y' = -rK(y - K) \). Solve this and explain what its solution tells us about solutions to the Gompertz equation.

Solution: We solve by separation: \( y'/(y - K) = -rK \), so that \( \ln|y - K| = -rKt + C' \). Exponentiating both sides, and letting \( C = \pm e^{C'} \), we have \( y = K + Ce^{-rKt} \). This says that if we start with an initial condition near \( y = K \), we expect the solution to converge to \( y = K \): that is, the equilibrium solution \( y = K \) is asymptotically stable.

c. [5 points] If we retain two terms from the Taylor expansion of \( \ln(K/y) \) near \( y = K \), we obtain the cubic differential equation \( y' = f(y) \), where \( f(y) \) is shown in the figure to the right. Sketch a phase line for this equation and explain what it suggests about the long-term behavior of the tumor.

Solution: Designating the critical point above \( y = K \) as \( y = y_1 \), we have the phase line shown below.

This suggests that for any initial condition \( y(0) = y_0 \) with \( 0 < y_0 < y_1 \), the solution will converge to \( y = K \) as \( t \to \infty \). However, if \( y_0 > y_1 \), the solution becomes unbounded as \( t \to \infty \).