6. [15 points] In lab 1 we considered the Gompertz equation, $y' = r\,y\,\ln(K/y)$. We explore this further in this problem.

a. [5 points] Consider the initial condition $y(0) = 1$. Find a linear approximation to the Gompertz equation that is valid near this initial condition. Under what conditions would you expect your approximation to be accurate?

Solution: Noting that $y' = r\,y(\ln(K) - \ln(y))$, we can use the Taylor expansion of $\ln(y) = 0 + (y-1) + \cdots$ to linearize the equation. To retain only linear terms, we truncate the log at the constant term 0, and so have $y' = r\ln(K)\,y$. This is a reasonable approximation, though a little bit of a fudge as we haven't expanded the y term in the equation.

If we are slightly more careful, we also expand the linear term y as $y = 1 + (y-1)$. Then $y' = r\,(1 + (y-1))(\ln(K) - (0 + (y-1)))$. To retain only terms in $(y-1)^0$ or $(y-1)^1$ we must truncate the expansion of the logarithm at the constant (0) term, so that $y' = r\,(1 + (y-1))(\ln(K)) = r\ln(K)\,y$.

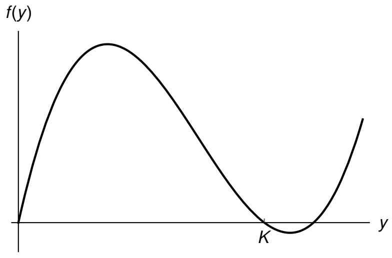
Alternately, if we expand the expression and then retain only linear terms in y, we obtain $y' = r\ln(K) + r\ln(K)(y-1) - r(1)(y-1) = r(\ln(K) - 1)y + r$. This retains a constant term r because $y = 1$ isn't a critical point of the equation.

In any case, this is valid when y is near 1, and as y moves away from that we would expect the approximation to rapidly get worse.

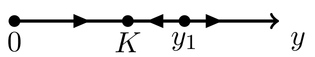
b. [5 points] We found that for y near K, the Gompertz equation is approximated as $y' = -rK(y-K)$. Solve this and explain what its solution tells us about solutions to the Gompertz equation.

Solution: We solve by separation: $y'/(y-K) = -rK$, so that $\ln|y-K| = -rKt + C'$. Exponentiating both sides, and letting $C = \pm e^{C'}$, we have $y = K + Ce^{-rKt}$. This says that if we start with an initial condition near $y = K$, we expect the solution to converge to $y = K$: that is, the equilibrium solution $y = K$ is asymptotically stable.

c. [5 points] If we retain two terms from the Taylor expansion of $\ln(K/y)$ near $y = K$, we obtain the cubic differential equation $y' = f(y)$, where $f(y)$ is shown in the figure to the right. Sketch a phase line for this equation and explain what it suggests about the long-term behavior of the tumor.



Solution: Designating the critical point above $y = K$ as $y = y_1$, we have the phase line shown below.



This suggests that for any initial condition $y(0) = y_0$ with $0 < y_0 < y_1$, the solution will converge to $y = K$ as $t \to \infty$. However, if $y_0 > y_1$, the solution becomes unbounded as $t \to \infty$.