7. [12 points] For each of the following, give an example as indicated, and a short (one or two sentence) explanation for how your example satisfies the indicated criteria.

a. [4 points] Give an example of an autonomous first-order differential equation with two equilibrium solutions, neither of which are stable.

Solution: An example is $y' = y^2(1-y)^2$. It has the two equilibrium solutions $y = 0$ and $y = 1$, but for all $y \neq 0, 1$, $y' > 0$. Thus neither is stable.

b. [4 points] Give an example of a linear, constant-coefficient system of two differential equations whose phase portrait is a stable counterclockwise spiral.

Solution: An example is $\mathbf{x}' = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \mathbf{x}$. We see that the eigenvalues are $\lambda = -1 \pm 2i$, so we have a stable spiral. To check that it is counterclockwise, we consider the direction of the trajectory at $(1, 0)$, where $\mathbf{x}' = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, that is, to the left and up, giving a counterclockwise spiral.

c. [4 points] Give an example of a linear, constant-coefficient system of two differential equations that has a critical point that is not at the origin.

Solution: All we need is that the system have a forcing, e.g., $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. The critical point of this is at $(1, -1)$.